ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 18, No. 1, 2011, pp. 33–53. © Pleiades Publishing, Ltd., 2011.

To the blessed memory of Vitaly Lazarevich Ginzburg

Spectral Series of the Schrödinger Operator in a Thin Waveguide with a Periodic Structure. 2. Closed Three-Dimensional Waveguide in a Magnetic Field

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Received January 31, 2011

Abstract. In the paper, which is the second part of the paper by J. Brüning, S. Dobrokhotov, S. Sekerzh-Zenkovich, T. Tudorovskiy, "Spectral series of the Schrödinger operator in thin waveguides with a periodic structure. 1," Russ. J. Math. Phys. **13** (4), 401–420 (2006), using the adiabatic approximation, diverse quantum states of the stationary Schrödinger equation for a particle in a thin waveguide in a magnetic field are constructed. The problems of "destruction" of the adiabatic approximation as the value of energy increases and of replacing this approximation by the approximation of V. P. Maslov's theory of complex germ (the complex WKB method) are studied.

DOI: 10.1134/S1061920811010055

1.THE STATEMENT OF THE PROBLEM: SCHRÖDINGER EQUATION FOR A THIN CLOSED WAVEGUIDE IN A MAGNETIC FIELD

In the paper, stationary states of a particle in a three-dimensional quantum thin toric waveguide with soft walls which is placed in a constant magnetic field are studied. We assume that the geometric axis of the waveguide is given by the equations $x_1 = \rho_0 \cos(\varphi), x_2 = \rho_0 \sin(\varphi), \zeta = 0$, where x_1, x_2, ζ dimensionless Cartesian coordinates on $\mathbb{R}^3$.

As in part I of the present paper (see [11]), we assume that the thickness of the waveguide is of the order of μ, where μ is supposed to be sufficiently small. In this case, the frequency of the (parabolic) confinement potential is of the order of μ^{-2}, and the potential is defined by the formula

$$v = \frac{1}{2\mu^4}\left[\Omega_1^2(\varphi,\mu)\left(\rho-\rho_0\right)^2 + \Omega_2^2(\varphi,\mu)\zeta^2\right].$$

We also assume that the magnetic field in the dimensionless variables is sufficiently large (in contrast to problems treated in [2]–[7]) and is of the order of μ^{-2}; thus, the value of this field is $\mu^{-2}H$, where $H \sim 1$. As far as the frequencies $\Omega_{1,2}$ of the confinement potential is concerned, we assume that these frequencies are given by the relations

$$\Omega_{1,2}(\varphi,\mu) = \Omega_{1,2}^{(0)} + \mu^{2\alpha}\Omega_{1,2}^{(1)}(\varphi), \tag{1.1}$$

where $0 \leqslant \alpha \leqslant 1$ and $\Omega_{1,2}^{(0)}(\varphi) > 0$ are some constants and $\Omega_{1,2}^{(1)}(\varphi)$ are smooth functions.

In the cylindrical coordinates ρ, φ, ζ with the axis ζ directed along the axis of the magnetic field, in the one-particle approximation, the wave function $\Psi/\sqrt{\rho}$ in the waveguide satisfies the three-dimensional Schrödinger equation

$$\frac{1}{2}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left(-i\frac{\partial}{\partial\varphi}+\frac{H\rho^2}{2\mu^2}\right)^2-\frac{\partial^2}{\partial\zeta^2}\right.$$
$$\left.+\frac{\Omega_1^2(\varphi,\mu)}{\mu^2}\left(\frac{\rho-\rho_0}{\mu}\right)^2+\frac{\Omega_2^2(\varphi,\mu)}{\mu^4}\zeta^2\right]\frac{\Psi}{\sqrt{\rho}}=E\frac{\Psi}{\sqrt{\rho}}, \tag{1.2}$$

where E stands for the energy of the corresponding quantum state. The function Ψ must be 2π-periodic with respect to the angular variable φ.

Introduce new variables y and z by defining them by the relations $\rho = \rho_0 + \mu y$ and $\zeta = \mu z$. Then equation (1.2) becomes

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial y^2}+\frac{1}{(\rho_0+\mu y)^2}\left(-i\mu\frac{\partial}{\partial\varphi}+\frac{H(\rho_0+\mu y)^2}{2\mu}\right)^2-\frac{\mu^2}{4(\rho_0+\mu y)^2}-\frac{\partial^2}{\partial z^2}\right.$$
$$\left.+\,\Omega_1^2(\varphi,\mu)y^2+\Omega_2^2(\varphi,\mu)z^2\right]\Psi=\mu^2E\Psi,\quad \Psi(\varphi+2\pi,y,z,\mu)=\Psi(\varphi,y,z,\mu). \tag{1.3}$$

To eliminate the parameter μ in the denominators of the coefficients of this equation, we pass from the function Ψ to the following new function Ψ':

$$\Psi=\exp\left(-\frac{iH\rho_0^2\varphi}{2\mu^2}\right)\Psi'. \tag{1.4}$$

Then we obtain the following equation for Ψ':

$$\hat{\mathcal{H}}\Psi'(\varphi,y,z,\mu)=\mu^2E\Psi'(\varphi,y,z,\mu), \tag{1.5}$$

where $\hat{\mathcal{H}}$ stands for the Schrödinger operator

$$\hat{\mathcal{H}}\equiv\frac{1}{2}\left[\frac{1}{(\rho_0+\mu y)^2}\left(-i\mu\frac{\partial}{\partial\varphi}+y\rho_0H+\frac{\mu}{2}y^2H\right)^2\right.$$
$$\left.-\frac{\partial^2}{\partial y^2}+\Omega_1^2(\varphi,\mu)y^2-\frac{\mu^2}{4(\rho_0+\mu y)^2}-\frac{\partial^2}{\partial z^2}+\Omega_2^2(\varphi,\mu)z^2\right]. \tag{1.6}$$

Since Ψ is a 2π periodic function of φ, it follows that Ψ' must satisfy the Bloch conditions

$$\Psi'(\varphi+2\pi,\rho,\zeta,\mu)=\exp\left(i\frac{\pi H\rho_0^2}{\mu^2}\right)\Psi'(\varphi,\rho,\zeta,\mu). \tag{1.7}$$

The objective of the paper is to construct asymptotic representations, for small μ, for the eigenvalues E and the eigenfunctions $\Psi(\rho,\varphi,\zeta,\mu)\in L_2(R^3)$ of the spectral problem (1.5)–(1.7).

Let us briefly describe the contents of the paper. In Sec. 2 generalizing results [11] and using the adiabatic approximation presented in a form of an "operator" separation of variables (see [2]–[7]) we reduce the original 3-D equation to 1-D equation on the axis of the wave guide. In Sec. 3 we discuss the situation when the variable thickness of the wave guide is changing very little which allow one to obtain the "limit long-wave" equation for the wave function on the axis of the wave guide. We give the semiclassical analysis along with the Reeb graph interpretation of asymptotic eigenfunctions of the reduced equation on the the axis in sec. 4. Here we discuss the so-called trapped modes and exited states corresponding to ballistic transport in the wave guide. The construction of the ultra-short wave functions is similar to the Born approximation, we discuss these functions in sec. 5. We apply to the original problem the Maslov complex germ (the complex WKB method) in the case when the adiabatic approximation stops working in sec. 6. Some cumbersome technical calculations are given in the appendix.

2. ADIABATIC APPROXIMATION

Since the functions $\Omega_1(\varphi,\mu)$ and $\Omega_2(\varphi)$ are smooth, we can apply the adiabatic approximation in the form of an "operator" separation of variables (see [2]–[7],[11]), to solve the three-dimensional (singularly perturbed) problem (1.5)–(1.7). Let us seek an approximate solution of the spectral problem (1.5)-(1.7) in the form

$$\Psi'(\varphi,y,z,\mu)=\hat{\chi}\psi'(\varphi,\mu). \tag{2.1}$$

Here $\hat{\chi}=\chi(-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},y,z,\mu)$ stands for a pseudodifferential operator and the function $\psi'(\varphi,\mu)$ is the solution of the one-dimensional spectral problem

$$\hat{\mathcal{L}}\psi'(\varphi,\mu)=\mu^2E\psi'(\varphi,\mu), \tag{2.2}$$

where

$$\hat{\mathcal{L}}=\mathcal{L}(-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},\mu) \tag{2.3}$$

stands for (in general, pseudodifferential) operator realizing the "Peierls substitution" (see [24]). As in part I, see [11], we use here the Feynman–Maslov notation and order.

Substituting (2.1) and (2.2) into (1.5) gives the operator equation

$$\hat{\mathcal{H}}\hat{\chi}=\hat{\chi}\hat{\mathcal{L}} \tag{2.4}$$

for the operators $\hat{\chi}$ and $\hat{\mathcal{L}}$.

Introduce the operator-valued symbol

$$\begin{aligned}\mathcal{H}=\frac{1}{2}\bigg[&\frac{1}{(\rho_0+\mu y)^2}\Big(p_\varphi+y\rho_0H+\frac{\mu}{2}y^2H\Big)^2\\&-\frac{\partial^2}{\partial y^2}+\Omega_1^2(\varphi,\mu)y^2-\frac{\mu^2}{4(\rho_0+\mu y)^2}-\frac{\partial^2}{\partial z^2}+\Omega_2^2(\varphi,\mu)z^2\bigg]\end{aligned} \tag{2.5}$$

for the Schrödinger operator

$$\hat{\mathcal{H}}=\mathcal{H}(-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},-i\partial/\partial y,y,-i\partial/\partial z,z,\mu),$$

and also the symbbol $\chi(p_\varphi,\varphi,y,z,\mu)$ for the operator $\hat{\chi}=\chi(-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},y,z,\mu)$ and the symbol $\mathcal{L}(p_\varphi,\varphi,\mu)$ for the operator $\hat{\mathcal{L}}=\mathcal{L}(-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},\mu)$. In this case, the operator equation (2.4) implies the equation

$$\mathcal{H}(p_\varphi-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},-i\partial/\partial y,y,-i\partial/\partial z,z,\mu)\chi(p_\varphi,\varphi,y,z,\mu)=\chi(p_\varphi-i\mu\overset{1}{\partial}/\partial\varphi,\overset{2}{\varphi},y,z,\mu)\mathcal{L}(p_\varphi,\varphi,\mu) \tag{2.6}$$

for the symbols $\chi(p_\varphi,\varphi,y,z,\mu)$ and $\mathcal{L}(p_\varphi,\varphi,\mu)$.

Let us solve the last equation by using a method of perturbation theory. The following expansion holds for $\mathcal{H}$:

$$\begin{aligned}\mathcal{H}(p_\varphi,\varphi,-i\partial/\partial y,y,-i\partial/\partial z,z,\mu)&=\mathcal{H}_0(p_\varphi,\varphi,-i\partial/\partial y,y,-i\partial/\partial z,z)\\&+\mu\mathcal{H}_1(p_\varphi,\varphi,-i\partial/\partial y,y,-i\partial/\partial z,z)+\mu^2\mathcal{H}_2(p_\varphi,\varphi,-i\partial/\partial y,y,-i\partial/\partial z,z)+\dots\end{aligned} \tag{2.7}$$

Introduce the notation

$$\Omega(\varphi,\mu)^2=\Omega_1^2(\varphi,\mu)+H^2,\qquad y_0=y_0(p_\varphi,\varphi)=r_0\frac{H^2}{\Omega^2(\varphi,\mu)},\qquad r_0=r_0(p_\varphi)=\frac{p_\varphi}{\rho_0H}.$$

Then it follows from (2.5) that

$$\mathcal{H}_0(p_\varphi,\varphi,-i\partial/\partial y,y,-i\partial/\partial z,z)$$
$$=\frac{1}{2}\left[\frac{p_\varphi^2}{\rho_0^2}\left(\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}\right)^2-\frac{\partial^2}{\partial y^2}+\Omega(\varphi,\mu)^2(y+y_0)^2-\frac{\partial^2}{\partial z^2}+\Omega_2^2(\varphi,\mu)z^2\right], \tag{2.8}$$
$$\mathcal{H}_1(p_\varphi,\varphi,-i\partial/\partial y,y)=-\frac{yp_\varphi^2}{\rho_0^3}-\frac{3y^2Hp_\varphi}{2\rho_0^2}-\frac{y^3H^2}{2\rho_0},\qquad \mathcal{H}_2(p_\varphi,\varphi,-i\partial/\partial y,y)|_{p_\varphi=0,\ H=0}=-\frac{1}{8\rho_0^2}.$$

We seek for the symbols $\chi(p_\varphi,\varphi,y,z,\mu)$ and $\mathcal{L}(p_\varphi,\varphi,\mu)$ in the form of asymptotic expansions as well,

$$\chi(p_\varphi,\varphi,y,z,\mu)=\chi_0(p_\varphi,\varphi,y,z)+\mu\chi_1(p_\varphi,\varphi,y,z)+\mu^2\chi_2(p_\varphi,\varphi,y,z)+\dots, \tag{2.9}$$
$$\mathcal{L}(p_\varphi,\varphi,\mu)=\mathcal{H}_{\text{eff}}(p_\varphi,\varphi)+\mu\mathcal{L}_1(p_\varphi,\varphi)+\mu^2\mathcal{L}_2(p_\varphi,\varphi)+\dots, \tag{2.10}$$

where, as in part I, the expression $\mathcal{H}_{\text{eff}}(p_\varphi,\varphi)$ stands for the leasing term of expansion (2.10).

Substituting (2.7)–(2.10) into (2.6) and equating the coefficients at like powers of μ, we obtain spectral problems for the corresponding symbols $\chi_0(p_\varphi,\varphi,y,z)$, $\mathcal{L}_0(p_\varphi,\varphi)$, $\chi_1(p_\varphi,\varphi,y,z)$, $\mathcal{L}_1(p_\varphi,\varphi)$, $\chi_2(p_\varphi,\varphi,y,z)$, and $\mathcal{L}_2(p_\varphi,\varphi)$. Strictly speaking, all these functions depend on the parameter μ, because $\Omega_{1,2}$ depends on μ (see (1.1)); we omit this dependence to simplify the notation.

We have already noted in part I [11] (see also [6] and [7], and this also becomes clear from the subsequent sections) that, to obtain the leading term of the asymptotic formula, one must find the first two summands in expansion (2.10) and, in some cases (for the "long waves") the symbol $\mathcal{L}_2(0,\varphi)$ as well.

Equating the terms of the order of $\mu^{(0)}$ in (2.6) gives spectral problems for $\chi_0(p_\varphi,\varphi,y,z)$ and $\mathcal{L}_0(p_\varphi,\varphi)$, namely,

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial y^2}+\Omega(\varphi,\mu)^2(y+y_0)^2-\frac{\partial^2}{\partial z^2}+\Omega_2^2(\varphi,\mu)z^2+\frac{p_\varphi^2}{\rho_0^2}\left(\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}\right)^2\right]\chi_0^{(\nu)}(p_\varphi,\varphi,y)$$
$$=\mathcal{H}_{\text{eff}}^{(\nu)}(p_\varphi,\varphi)\chi_0^{(\nu)}(p_\varphi,\varphi,y). \tag{2.11}$$

Here $\nu=(\nu_1,\nu_2)$ (ν_1 and ν_2 stand for the quantum numbers corresponding to the two "fast variables," y and z).

The solutions of (2.11) are

$$\mathcal{H}_{\text{eff}}^{(\nu)}(p_\varphi,\varphi)=\frac{p_\varphi^2}{2\rho_0^2}\left(\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}\right)^2+v_{\text{eff}}^{(\nu)},\quad v_{\text{eff}}^{(\nu)}=\Omega(\varphi,\mu)(\nu_1+1/2)+\Omega_2(\varphi,\mu)(\nu_2+1/2), \tag{2.12}$$
$$\chi_0^{(\nu)}(p_\varphi,\varphi,y,z)=\Omega(\varphi,\mu)^{1/4}\Omega_2(\varphi)^{1/4}\zeta^{(\nu_1)}\Big(\sqrt{\Omega(\varphi,\mu)}\big(y+y_0(p_\varphi,\varphi)\big)\Big)\zeta^{(\nu_2)}(\sqrt{\Omega_2(\varphi,\mu)}z), \tag{2.13}$$

where

$$\zeta^{(k)}(\xi)=\frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2^kk!}}\exp(-\xi^2/2)H_n(\xi)$$

provided that $H_n(\xi)$ are the corresponding hermite polynomials.

Note that the functions $\chi_0^{(\nu)}$ *depend on* p_φ *by means of* $y_0(p_\varphi,\varphi)$ not polynomially, and thus the leading term $\hat{\chi}$ is a *pseudodifferential operator* indeed, in contrast to the similar operator in part I (see [11]).

The manipulations that are quite similar to those used in [11] (see Appendix A) give

$$\mathcal{L}_1^{(\nu)}(p_\varphi,\varphi) = \frac{\Omega_1^2(\varphi,\mu)}{\Omega^4(\varphi,\mu)}\left(1-\frac{H^2}{2\Omega^2(\varphi,\mu)}\right)\frac{Hp_\varphi^3}{\rho_0^4} - \frac{3H(\nu_1+1/2)\Omega_1^2(\varphi,\mu)}{2\rho_0^2\Omega^3(\varphi,\mu)}p_\varphi$$
$$-i\frac{\partial}{\partial\varphi}\left(\frac{\Omega_1^2(\varphi,\mu)}{\Omega^2(\varphi,\mu)}\right)\frac{p_\varphi}{2\rho_0^2}, \tag{2.14}$$

The evaluation of the symbol $\mathcal{L}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}$ without additional assumptions is a rather complicated problem technically. Additional assumptions occur for the case in which the problem under consideration admits "long-wave" regimes (see part I in [11]). When studying regimes described by rapidly oscillating functions, the consideration of the symbol $\mu^2\mathcal{L}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}$ gives only a small correction to the leading term of the asymptotic expansion, and therefore we return to the evaluate of this symbol in the section devoted to the "long-wave" regime.

Thus, the "essential" part of the symbol $\mathcal{L}(p_\varphi,\varphi,\mu)$ is

$$\mathcal{L}^{(\nu)}(p_\varphi,\varphi,\mu) = \mathcal{H}_{\text{eff}}^{(\nu)}(p_\varphi,\varphi) + \mu\mathcal{L}_1^{(\nu)}(p_\varphi,\varphi) + \mu^2\mathcal{L}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}. \tag{2.15}$$

Applying now the procedure of "quantization," we define the operators $\hat{\mathcal{H}}_{\text{eff}}^{(\nu)}$, $\hat{\mathcal{L}}_1^{(\nu)}$, and $\hat{\mathcal{L}}_1^{(\nu)}$ by replacing the variable p_φ in the symbols (2.12), (2.14), and (2.15) by the operators $-i\mu\partial/\partial\varphi$. Further, we construct the operator $\hat{\mathcal{L}}^{(\nu)}$ by using equation (2.15) and by substituting $\hat{\mathcal{L}}^{(\nu)}$ into (2.2) instead of $\hat{\mathcal{L}}$. Finally, taking (1.5) and (2.1) into account, we arrive at the following spectral one-dimensional problem:

$$\left[\hat{\mathcal{H}}_{\text{eff}}^{(\nu)}(p_\varphi,\varphi) + \mu\hat{\mathcal{L}}_1^{(\nu)}(p_\varphi,\varphi) + \mu^2\hat{\mathcal{L}}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}\right]\psi^{(\nu,n)} = \mu^2E^{(\nu,n)}\psi^{(\nu,n)}, \tag{2.16}$$

$$\psi^{(\nu,n)}(\varphi+2\pi) = \exp\left(i\frac{\pi H\rho_0^2}{\mu^2}\right)\psi^{(\nu,n)}(\varphi), \tag{2.17}$$

for $E^{(\nu,n)}$ and $\psi^{(\nu,n)}$ in L_2 indexed by three-dimensional quantum numbers $\nu=(\nu_1,\nu_2),n$.

If a solution (2.16), (2.17) is obtained, then the corresponding (asymptotic) solution $\Psi^{(\nu,n)}$ of equation (1.3) can be recovered from $\psi^{(\nu,n)}$ by the following

$$\Psi^{(\nu,n)} = \exp\left(-i\frac{H\rho_0^2\varphi}{2\mu^2}\right)(\hat{\chi}_0^{(\nu)} + \mu\hat{\chi}_1^{(\nu)} + \mu^2\hat{\chi}_1^{(\nu)})\psi^{(\nu,n)},\ \hat{\chi}_j^{(\nu)} = \chi_j^{(\nu)}(-i\mu\overset{1}{\partial}/\partial x, \overset{2}{x}, y, z,). \tag{2.18}$$

The function $\chi_0^{(\nu)}$ is defined in (2.13), whereas the functions $\chi_1^{(\nu)}$ and $\chi_2^{(\nu)}$ are defined in Appendix B.

It is clear that the magnetic field, together with the confinement potential, defines a metric on the line $\mathbb{R}_\varphi$ (or an effective mass depending on φ) given by the factor $\Omega^2(\varphi,\mu)/\Omega_1^2(\varphi,\mu)$, and also replaces the periodicity conditions by the Bloch conditions (2.17).

3. LIMIT LONG-WAVE EQUATION

Similarly to [11], the "long-wave" regimes (cf. [12], [16], [25], [19], and [33]) exist only if the modification of the thickness of the waveguide is small, namely, under the assumption that

$$\Omega_k(\varphi) = \Omega_k^{(0)} + \mu^2\tilde{\Omega}_k(\varphi), \quad \Omega_k^{(0)} = \text{const} > 0, \quad k=1,2, \tag{3.1}$$

and thus

$$\Omega(\varphi,\mu) = \Omega^{(0)} + \mu^2\Omega^{(1)}(\varphi), \quad \Omega^{(0)} = \sqrt{\Omega_1^{(0)} + H^2}, \quad \Omega^{(1)}(\varphi) = \frac{\Omega_1^{(0)}}{\Omega^{(0)}}\tilde{\Omega}_1(\varphi).$$

Neglecting the summands of the order of $o(\mu^2)$, we can write out the operators generated by the symbols (2.12) and (2.14) in the form

$$\mathcal{H}_{\rm eff}^{(\nu)} \simeq \Omega^{(0)}(\nu_1+1/2) + \Omega_2^{(0)}(\nu_2+1/2) \\ +\mu^2\left[-\frac{1}{2\rho_0^2}\left(\frac{\Omega_1^{(0)}}{\Omega^{(0)}}\right)^2\frac{\partial^2}{\partial\varphi^2} + \Omega^{(1)}(\varphi)(\nu_1+1/2) + \tilde{\Omega}_2(\varphi)(\nu_2+1/2)\right] + o(\mu^2),$$

$$\mu\hat{\mathcal{L}}_1^{(\nu)} \simeq \mu^2 i\frac{3HE_{\nu_1}\Omega_1^{(0)}}{2\rho_0^2\Omega^{(0)2}}\frac{\partial}{\partial\varphi} + o(\mu^2).$$

The evaluation of the quantity $\mathcal{L}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}$ turns out to be not quite trivial, and the corresponding manipulations (presented in Appendix 2 give

$$\mu^2\hat{\mathcal{L}}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0} \simeq \mu^2\left\{\frac{H^2}{32\rho_0^2\Omega^{(0)2}}(30\nu_1(\nu_1+1)\Omega_1^{(0)} + 15\Omega^0 - 11H^2) - \frac{1}{8\rho_0^2}\right\} + o(\mu^2).$$

The substitution of these expressions into (2.16) and the renormalization of the energy $E^{(\nu,n)}$ in the form

$$\tilde{E}^{(\nu,n)} = E^{(\nu,n)} - \frac{1}{\mu^2}[\Omega^{(0)}(\nu_1+1/2) + \Omega_2^{(0)}(\nu_2+1/2)] \\ -\frac{H^2}{32\rho_0^2\Omega^{(0)4}}\Big(30\nu_1(\nu_1+1)\Omega_1^{(0)2} + 15\Omega^{(0)2} - 11H^2\Big) + \frac{1}{8\rho_0^2} \tag{3.2}$$

lead to the equation

$$\left[-\frac{1}{2\rho_0^2}\left(\frac{\Omega_1^{(0)}}{\Omega^{(0)}}\right)^2\frac{\partial^2}{\partial\varphi^2} + i\frac{3HQ_{\nu_1}\Omega_1^{(0)2}}{2\rho_0^2\Omega^{(0)4}}\frac{\partial}{\partial\varphi} + \Omega^{(1)}(\varphi)(\nu_1+1/2) + \tilde{\Omega}_2(\varphi)(\nu_2+1/2)\right]\psi^{(\nu,n)} = \tilde{E}^{(\nu,n)}\psi^{(\nu,n)}.$$

The last equation can be transformed into the following one-dimensional Schrödinger equation with periodic coefficients (or into a Hill equation)

$$\left[-\frac{1}{2\rho_0^2}\left(\frac{\Omega_1^{(0)}}{\Omega^{(0)}}\right)^2\frac{\partial^2}{\partial\varphi^2} + \Omega^{(1)}(\varphi)(\nu_1+1/2) + \tilde{\Omega}_2(\varphi)(\nu_2+1/2)\right]\breve{\psi}^{(\nu,n)} = \breve{E}^{(\nu,n)}\breve{\psi}^{(\nu,n)} \tag{3.3}$$

for the function

$$\breve{\psi}^{(\nu,n)} = \psi^{(\nu,n)}e^{-i\alpha\varphi}, \quad \alpha = \frac{3HQ_{\nu_1}}{2\Omega^{(0)2}}, \quad Q_{\nu_1} = \frac{\Omega_1^{(0)}(\nu_1+1/2)}{\Omega^{(0)}}, \tag{3.4}$$

where

$$\breve{E}^{(\nu,n)} = \tilde{E}^{(\nu,n)} + \frac{9}{8}\left(\frac{HQ_{\nu_1}}{\Omega^{(0)2}}\right)^2. \tag{3.5}$$

The function $\tilde{\psi}^{(\nu,n)}$ must satisfy the Bloch conditions (see (2.17) and (3.4)),

$$\breve{\psi}^{(\nu,n)}(\varphi+2\pi) = \exp\left[i\pi\left(\frac{H\rho_0^2}{\mu^2} - 2\alpha\right)\right]\breve{\psi}^{(\nu,n)}(\varphi). \tag{3.6}$$

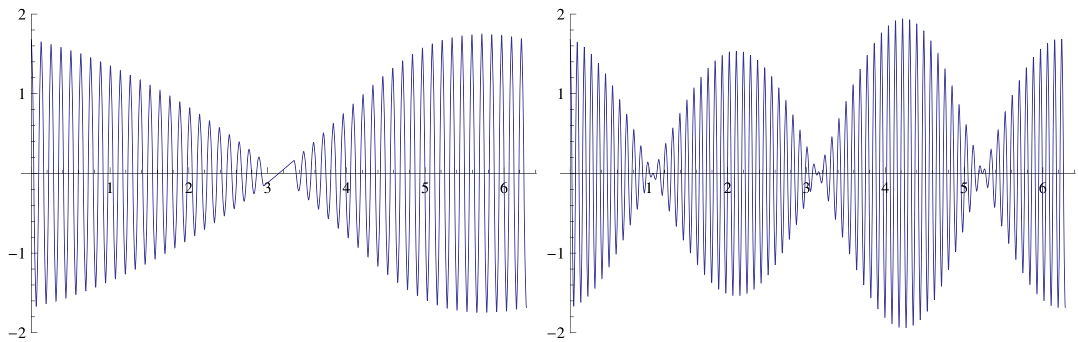

Fig. 1. Real parts of the wave functions on the axis of the waveguide of the original Schrödinger operator that correspond to long-wave regimes. The case of a single oscillation of a reduced equation is presented to the left, and the case in which there are three oscillations of this kind is presented to the right.

As is well known ([17], [18], [21], [32], and see also [10]), the spectrum of problem (3.3), (3.6) in $L_2(\mathbb{R}_\varphi)$ is determined by a family of dispersion curves $\mathcal{E} = \mathcal{E}^{(\nu,n)}(k)$ on the $(k,\mathcal{E})$-plane, where k stands for the quasimomentum, $\mathcal{E}$ for the energy, and n for the index of the zone. We may assume here that $\pi n \leqslant k \leqslant \pi(n+1)$ in the nth zone. Denote the corresponding Bloch function by $\vartheta^{(\nu,n)}(\varphi,k)$. It is clear that, to find a solution of problem (3.3), (3.6), one should set

$$k = k_n \equiv \pi\Big(\frac{H\rho_0^2}{\mu^2} - 2\alpha\Big) + 2\pi m(n),$$

where $m(n)$ is an integer such that

$$n \leqslant \Big(\frac{H\rho_0^2}{\mu^2} - 2\alpha\Big) + 2m(n) \leqslant n+1.$$

This gives an eigenfunction

$$\breve{\psi}^{(\nu,n)} = \vartheta^{(\nu,n)}\left(\varphi, k_n\right). \tag{3.7}$$

Using (3.2) and (3.5), we see that

$$\begin{aligned} E^{(\nu,n)} = {} & \frac{1}{\mu^2}\Big[\Omega^{(0)}(\nu_1+1/2) + \Omega_2^{(0)}(\nu_2+1/2)\Big] \\ & + \mathcal{E}^{(\nu,n)}\left(k_n\right) + \frac{H^2}{32\rho_0^2\Omega^{(0)4}}\Big(30\nu_1(\nu_1+1)\Omega_1^{(0)2} + 15\Omega^{(0)2} - 11H^2\Big) - \frac{1}{8\rho_0^2}. \end{aligned} \tag{3.8}$$

Finally, we arrive at the following assertion.

Proposition 3.1. *Let the functions* $\Omega_{1,2}$ *be of the form* (3.1); *let the functions* $\breve{\psi}^{(\nu,n)}$ *and the quantities* $E^{(\nu,n)}$ *be defined by* (3.7) *and* (3.8). *In this given by relations* (1.4) *and* (2.18) *satisfy the original equation* (1.3) *up to* $O(\mu^3)$, *and the numbers* $E^{(\nu,n)}$ *approximate some eigenvalues of the original operator* mod $O(\mu^3)$.

Note that, if μ, H, and ρ_0 are fixed, the expression $\frac{H\rho_0^2}{\mu^2} - 2\alpha$ turns out to be large, whereas the quasimomentum k_n is chosen to be finite. Recall that the number of oscillations of the real are imaginary parts of the Bloch functions in question on the interval $[0,2\pi]$ is equal to n, whereas the Bloch function by itself can be represented in the form

$$\vartheta^{(\nu,n)}\left(\varphi,k_n\right) = e^{ik_n\varphi}\Theta^{(\nu,n)}\left(\varphi,k_n\right),$$

where $\Theta(\nu, n)\,(\varphi, k_n)$ stands for a nonvanishing 2π-periodic function of φ. Thus, the complete wave function (1.4) has a rapidly oscillating factor

$$\exp\left\{i\left(k_n - \frac{H\rho_0^2\varphi}{2\mu^2}\right)\varphi\right\},$$

and the quasimomentum k_n is only a small correction to the wave number along the axis of the waveguide. The function $\Theta^{(\nu,n)}\,(\varphi, k_n)$ determines smooth varieties of the amplitude of the complete wave function along the variable φ (see Fig. 1). We also see that, a shift of the spectrum of the operator in question arises; this shift is generated by the magnetic flow through the area bounded by the closed axis of the waveguide. The flow thus occurring is a consequence of the so-called *Aharonov–Bohm effect*. However, this effect consists of two parts. The correction $-\alpha$ corresponds to the displacement of the "actual" axis of the waveguide, namely, it is moved somewhat away from the circle $\rho = \rho_0, z = 0$, and therefore the actual area of the domain bounded by the axis of the waveguide, as well as the magnetic flow through the domain, are to be corrected, which gives the additional summand $-\alpha$.

4. SEMICLASSICAL ASYMPTOTIC RELATIONS

Let us now consider waveguides given by confinement potentials of the form

$$\Omega_{1,2}(\varphi, \mu) = \Omega_{1,2}^{(0)} + \mu^{2\alpha}\Omega_{1,2}^{1}(\varphi), \qquad 1 > \gamma \geqslant 0.$$

After a corresponding re-expansion and a modification of the wave function we obtain a spectral problem for the one-dimensional periodic Schrödinger operator with the Bloch conditions including the magnetic flow generated by the Aharonov–Bohm effect and entering the phase factor. Similarly to the previous case, this leads to a sift of the spectrum, and the shift contains the magnetic flow. All these effects manifest themselves for the case in which $\Omega_{1,2}(\varphi, \mu)$ depends on φ rather strongly, i.e., under the assumption that the derivatives $\frac{\partial\Omega_{1,2}}{\partial\varphi}$ are not small as compared with the parameter μ. In our subsequent investigations, we dwell on this very situation in detail.

In the case under consideration, we may apply the semiclassical approximation. The classical Hamiltonian is of the form (2.12), and it can be treated as a Hamiltonian on a circle with nonconstant metric.A simple change of variables enables us to reduce the problem with the Hamiltonian (2.12) to a problem with the Hamiltonian with constant metric; this corresponds to "straightening" the metric in the principal symbol of equation (2.16). Namely, denote by K the mean value

$$K = \frac{1}{2\pi}\int_0^{2\pi} \frac{\Omega(\varphi, \mu)}{\Omega_1(\varphi, \mu)} d\varphi \tag{4.1}$$

and write

$$\phi(\varphi) = \frac{1}{K}\int_0^{\varphi} \frac{\Omega(\varphi, \mu)}{\Omega_1(\varphi, \mu)} d\varphi. \tag{4.2}$$

It can readily be seen that the function $\phi(\varphi)$ can be represented in the form

$$\phi(\varphi) = \varphi + \phi_0(\varphi), \tag{4.3}$$

where $\phi_0(\varphi)$ is a smooth 2π-periodic function. Using (4.2), we immediately see that there is a smooth function $\varphi(\phi)$ inverse to $\phi(\varphi)$. Applying (4.3) shows that $\varphi(\phi)$ is representable in the form similar to (4.3),

$$\varphi(\phi) = \phi + \varphi_0(\phi), \tag{4.4}$$

where $\varphi_0(\phi)$ is a smooth 2π-periodic function.

Let us now pass in equation (2.16) from the variable φ to the variable ϕ. The derivative becomes $\frac{\partial}{\partial\varphi} = g(\phi)\frac{\partial}{\partial\phi}$, where

$$g(\phi) = \frac{\partial\phi(\varphi)}{\partial\varphi}\Big|_{\varphi=\varphi(\phi)} = \frac{1}{K}\frac{\Omega(\varphi, \mu)}{\Omega_1(\varphi, \mu)}\Big|_{\varphi=\varphi(\phi)},$$

and the operator $\hat{\mathcal{L}}$ acquires the form $\mathcal{L}(\overbrace{g(\phi)\frac{\partial}{\partial\phi}}^{1}, \varphi(\overset{2}{\phi}), \mu)$. Let us also "correct" the functions $\psi^{(\nu,n)}$ by setting $\psi^{(\nu,n)} = \sqrt{g}\check{\psi}^{(\nu,n)}$. In this case, with regard to the formulas

$$\frac{\partial^2}{\partial\varphi^2} = g^2\frac{\partial^2}{\partial\phi^2} + gg_\phi\frac{\partial}{\partial\phi}$$

and

$$\frac{\partial^2}{\partial\varphi^3} = g^3\frac{\partial^3}{\partial\phi^3} + 3g^2g_\phi\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}\frac{\partial g^2}{\partial\phi}\frac{\partial}{\partial\phi},$$

the symbol of the operator $\frac{1}{\sqrt{g}}\mathcal{L}(\overbrace{g(\phi)\frac{\partial}{\partial\phi}}^{1}, \varphi(\overset{2}{\phi}), \mu)\sqrt{g}$ (up to summands $O(\mu^2)$ which give only a small correction to the leading term of the asymptotic expansion) becomes

$$\check{\mathcal{H}}_{\text{eff}}^{(\nu)}(p_\varphi, \varphi) + \mu\check{\mathcal{L}}_1^{(\nu)}(p_\varphi, \varphi) + O(\mu^2), \tag{4.5}$$

where

$$\check{\mathcal{H}}_{\text{eff}}^{(\nu)} = \frac{1}{2\rho_0^2K^2}p_\phi^2 + \check{v}_{\text{eff}}^{(\nu)}, \quad \check{v}_{\text{eff}}^{(\nu)} = \Omega(\varphi,\mu)(\nu_1 + 1/2) + \Omega_2(\varphi,\mu)(\nu_2+1/2)|_{\varphi=\varphi(\phi)}, \tag{4.6}$$

$$\check{\mathcal{L}}_1^{(\nu)}(p_\varphi,\varphi) = \left[\frac{H}{K^3\rho_0^4\Omega_1(\varphi,\mu)\Omega(\varphi,\mu)}\left(1 - \frac{H^2}{2\Omega^2(\varphi,\mu)}\right)p_\phi^3 - \frac{3H(\nu_1+1/2)\Omega_1(\varphi,\mu)}{2K\rho_0^2\Omega^2(\varphi,\mu)}p_\phi\right]\Big|_{\varphi=\varphi(\phi)}. \tag{4.7}$$

The effective Hamiltonian (the principal symbol) acquires the standard form for the one-dimensional Schrödinger equation with periodic potential, and therefore practically all considerations of Sec. 2 in [10] concerning the semiclassical analysis of the spectral problem for the operator

$$\hat{H} = -h^2\partial^2/\partial x^2 + V(x), \qquad h << 1,$$

with 2π-periodic smooth potential $V(x)$ remain valid for (2.16). The difference is in taking into account the subprincipal symbol $\check{\mathcal{L}}_1^{(\nu)}$, which leads to the modification of the amplitude and of the energy levels evaluated according to the standard formulas in [28]. One can carry out the corresponding considerations both for the equation with "straightened metric" and for the original equation (2.16), although, certainly, it is more reasonable to present the final formulas in terms of the original angular variable, and we shall proceed in this way below.

Thus, consider diverse quantum states (regimes described by the spectral problem (2.16)–(2.17)). The corresponding critical points on the plane (p_φ, φ) are $p_\varphi = 0$, $\varphi = \varphi_0^{(\nu)}$, where $\varphi_0^{(\nu)}$ are the critical points of the effective potential $v_{\text{eff}}^{(\nu)}(\varphi)$ introduced in (2.12). The phase portrait for the trajectories of the Hamiltonian system with the Hamiltonian (2.12) repeats the standard phase portrait for the Hamiltonian

$$\frac{1}{2m}p_\varphi^2 + \check{v}_{\text{eff}}^{(\nu)}(\phi), \qquad m = K^2\rho_0^2.$$

The critical points are the same, and the structure of the separation of trajectories into parts formed by closed and nonclosed trajectories separated by separatrices are analogous. This leads to the spectrum of the reduced equation which is similar to the spectrum described in Sec. 2 of part 1 [11]; however, some constructions arise due to the new terms (as compared with formula (2.2) of part 1) and, which is more important, to the influence of the magnetic field on the spectrum. Let us briefly describe the corresponding asymptotic eigenvalues and eigenfunctions. Assyume that the function $v_{\text{eff}}^{(\nu)}(\varphi)$ has only one nondegenerate minimum $\varphi_0^{(\nu)}$ and one nondegenerate maximum $\varphi_1^{(\nu)}$

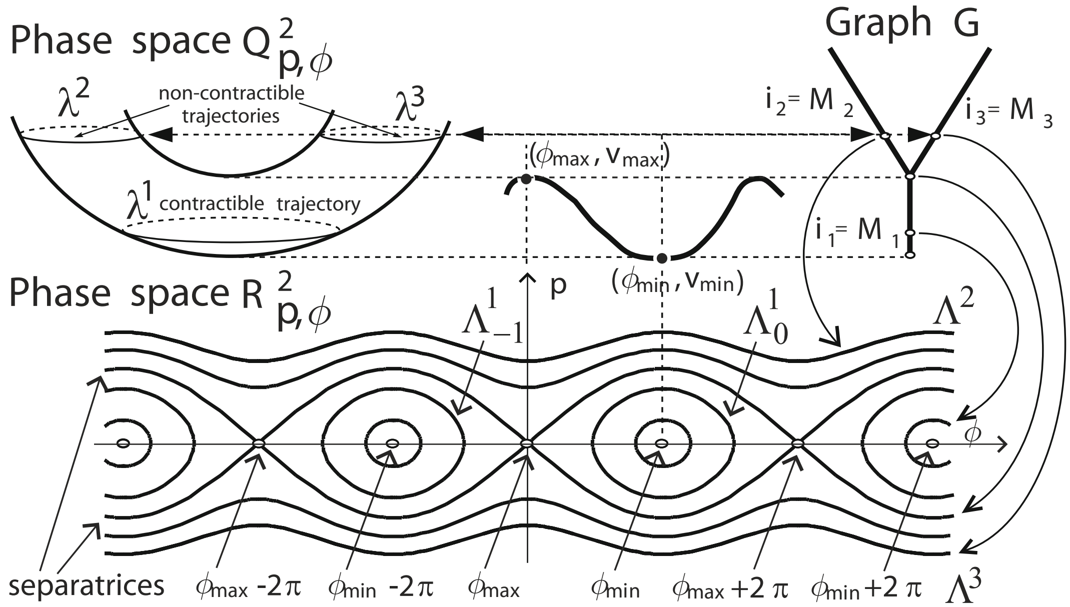

Fig. 2. In the upper figure we show the phase space in the form of a "deformed cylinder" and indicate closed trajectories on this cylinder that are classified by using points on the edges of the Reeb graph (above and to the right). In the lower figure we show trajectories (the phase portrait) of the same trajectories which are now drawn on the phase plane $\mathbb{R}^2_{p_\varphi\varphi}$. To trajectories on the cylinder there correspond closed trajectories on $\mathbb{R}^2_{p_\varphi\varphi}$, whereas nonclosed trajectories correspond to noncontractible trajectories on the cylinder. The contractible and noncontractible trajectories are separated by separatrices.

on the interval $[0, 2\pi]$. Note that, in contrast to the example in Sec. 2, the points $\varphi_0^{(\nu)}$ and $\varphi_1^{(\nu)}$ depend on ν. Write $V^{(\nu)}_{\min} = v^{(\nu)}_{\text{eff}}(\varphi_0^{(\nu)})$ and $V^{(\nu)}_{\max} = v^{(\nu)}_{\text{eff}}(\varphi_1^{(\nu)})$.

To make our exposition more complete, we present the corresponding figure (see Fig. 2) and the comments of [10] related to this simplest case. Since $v^{(\nu)}_{\text{eff}}(\varphi)$ is periodic, it follows that the phase space of the Hamiltonian system corresponding to (2.12) can also conveniently be represented as a "deformed" cylinder (see Fig. 2) in a space with vertical axis showing the values of the classical effective Hamiltonian (the energy). The sections of this cylinder that correspond to diverse values of energy give diverse closed curves (trajectories) on the cylinder $Q^2_{p_\varphi\varphi}$ to which closed and nonclosed trajectories on the Euclidean phase plane correspond. It is appropriate to characterize each of these trajectories by points on the edges of the Reeb graph G (which is a topological characteristic of the Morse function) of the effective Hamiltonian (see [8] and [10]). To the minimal value of the effective Hamiltonian $V^{(\nu)}_{\min}$ there corresponds a point at the "bottom" of the cylinder, the rest points (factorized $\bmod 2\pi$) on the Euclidean phase plane $\mathbb{R}^2_{p_\varphi\varphi}$, and the beginning of the edge i_1 of the Reeb graph G. To the energies in the interval $(V^{(\nu)}_{\min}, V^{(\nu)}_{\max}$ there correspond contractible trajectories on the cylinder $Q^2_{p_\varphi\varphi}$, closed trajectories on the phase plane $\mathbb{R}^2_{p_\varphi\varphi}$ (factorized $\bmod 2\pi$), and the interior points of the edge i_1 of the Reeb graph G. To the value of energy $V^{(\nu)}_{\max}$ there corresponds the upper endpoint of the edge i_1 and the lower endpoints of the edges $i_{2,3}$ of the graph G, and also the trajectories in the form of "figure-of-eight" on the cylinder $Q^2_{p_\varphi\varphi}$ and the separatrices on the plane $\mathbb{R}^2_{p_\varphi\varphi}$. After the effective Hamiltonian passes the value $V^{(\nu)}_{\max}$, two curves (still closed but already noncontractible) occur in the cylinder $\mathbb{R}^2_{p_\varphi\varphi}$, and two infinite trajectories on the plane $\mathbb{R}^2_{p_\varphi\varphi}$ (whose momenta p_φ are periodic with respect to φ) correspond to these curves. Each of these trajectories is characterized by a point on the edge i_2 of the graph G (the lower curve) and a point on the edge i_3 of G (the upper curve). It is natural to use action variables as quantitative characteristics. These

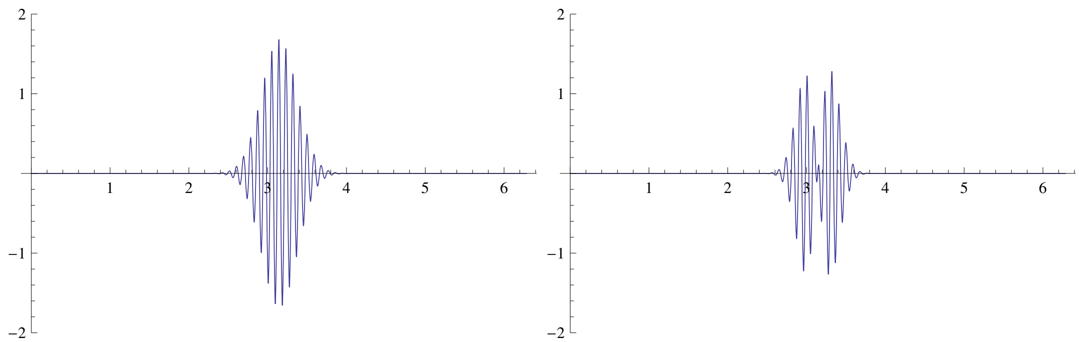

Fig. 3. The real parts of the wave functions on the axis of the waveguide of the original Schrödinger operator that correspond to trap states in the lower underbarrier domain. The case of a single oscillation of the reduced equation is shown to the left, and the case in which there are two such oscillations is presented to the right. On the Reeb graph, to these states there correspond the lower points of the edge i_1 and neighborhoods of the bottom of the deformed phase cylinder, i.e., of the rest points on the phase plane in Fig. 2.

variables are as follows: the variable I^1 equal to the area, of a closed trajectory on $\mathbb{R}^2_{p_\varphi\varphi}$, divided by 2π, and the variables I^2 and I^3 that are equal to the areas of curvilinear trapezia, bounded by nonclosed curves on $\mathbb{R}^2_{p_\varphi\varphi}$ and segments of vertical lines $\varphi = 0, \varphi = 2\pi$, divided by 2π. Thus, the action variables I^j define a one-to-one correspondence between the points of the Reeb graph G and the values of the effective Hamiltonian $\mathcal{H}^{(\nu)}_{\text{eff}} = \mathcal{H}^{(\nu)}_{\text{eff}}(I^j)$, $j = 1, 2, 3$. When constructing asymptotic eigenfunctions, the values I^j range over a discrete set (these are quantized by the Bohr–Sommerfeld rule), which gives a family of points on the graph G; their projection to the energy axis gives a part of the spectrum corresponding to the index ν of the "subzone of dimensional quantization" of the waveguide.

4.1. Trap states (trapped modes). The lower underbarrier domain. Similarly to Sec. 2 of part 1, eigenvalues are localized in a neighborhood of $V^{(\nu)}_{\text{min}}$, which also means that the numbers $n\mu$ are sufficiently small; to these values there correspond the lower points of the edge i_1 of the Reeb graph G. In this case, one can use the approximation of harmonic oscillator and write

$$E^{(\nu,n)} = v^{(\nu)}_{\text{eff}}(\varphi^{(\nu)}_0) + \mu(n + 1/2)\omega_0 + O(\mu^2), \quad \omega_0 = \rho_0 \left(\frac{\Omega(\varphi^{(\nu)}_0)}{\Omega_1(\varphi^{(\nu)}_0)} \right) \sqrt{2 \frac{\partial^2 v^{(\nu)}_{\text{eff}}}{\partial \varphi^2}(\varphi^{(\nu)}_0)}. \tag{4.8}$$

The wave functions are localized in a neighborhood of $\varphi^{(\nu)}_0 + 2\pi k$, $k = 0, \pm 1, \dots$, and these functions have the following form in a neighborhood of the point $\varphi^{(\nu)}_0$:

$$\psi^{(\nu,n)}(\varphi) = C^{(n)} \exp(-\xi^2/2) H_n(\xi), \quad \xi = \sqrt{\omega_0}(\varphi - \varphi^{(\nu)}_0), \tag{4.9}$$

where $C^{(n)}$ stands for a normalizing constant. In order to obtain functions $\psi^{(\nu,n)}(\varphi)$ satisfying the Bloch condition (2.17) on $\mathbb{R}_\varphi$, one should extend this function by setting (see, e.g., [10])

$$\psi^{(\nu,n)}(\varphi) = \exp\left(ik \frac{\pi H \rho_0^2}{\mu} \right) \psi^{(\nu,n)}(\varphi - 2\pi k), \quad \varphi \in [2\pi k, 2\pi(k+1)]. \tag{4.10}$$

The number n corresponds to the number of oscillations on the period $[0, 2\pi]$. The influence of the magnetic field on the eigenvalues is in the dependence of the frequency ω_0 on H. The effect related to the occurrence of a nontrivial quasimomentum is negligibly small, as in the case of long-wave limit (se (3.2)) .

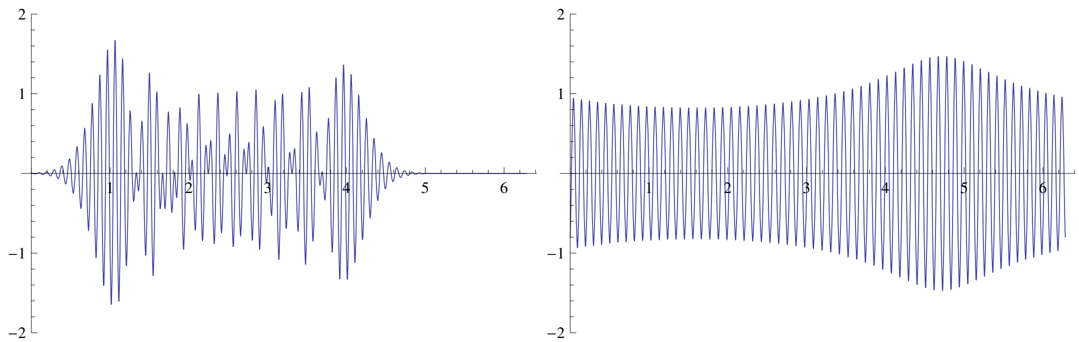

Fig. 4. To the left, we present the real parts of the wave functions on the axis of the waveguide of the original Schrödinger operator that correspond to trap states in the underbarrier domain. On the Reeb graph, to these states there correspond the points placed inside the edge i_1 and contractible curves of the deformed phase cylinder, i.e., closed curves on the phase plane in Fig. 2. To the right, we show the real parts of the wave functions on the axis of the waveguide of the original Schrödinger operator that correspond to the excited states of the overbarrier domain, i.e., to the so-called "ballistic transport" in a waveguide. On the Reeb graph, to these states there correspond points placed at the edges i_2, i_3 and the noncontractible curves on the deformed phase cylinder, i.e., the nonclosed curves on the phase plane in Fig. 2.

4.2. Excited trap states (trapped modes). In this case, one can use the standard semiclassical approximation. As in Sec. 2 of part 1, consider the case in which $n \sim \mu^{-1}$. To this situation, there correspond internal points of the edge i_1 of the Reeb graph G.

For $E^{(\nu,n)} < v_{\max} - \delta$ and $n \sim \mu^{-1}$, we have

$$E^{(\nu,n)} = \mathcal{E}_{\mathrm{un}}^{(\nu,n)} + \mu\Lambda^{(\nu,n)} + o(\mu), \tag{4.11}$$

where $\mathcal{E}_{\mathrm{un}}^{(\nu,n)}$ are defined from the *Bohr–Sommerfeld quantization condition*

$$\frac{\rho_0}{\pi}\int_{\varphi_-}^{\varphi_+}\frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)}\sqrt{\mathcal{E}_{\mathrm{un}}^{(\nu,n)} - v_{\mathrm{eff}}^{(\nu)}(\varphi,\mu)}\,d\varphi = \mu(n+1/2), \tag{4.12}$$

$\varphi_\pm$ are solutions of the equations $v_{\mathrm{eff}}^{(\nu)}(x,\mu) = \mathcal{E}_{\mathrm{un}}^{(\nu,n)}$, and

$$\Lambda_{\mathrm{un}}^{(\nu,n)} = \frac{1}{2\pi}\int_{\varphi_-}^{\varphi_+}\Big(\frac{3H(\nu_1+1/2)}{\Omega(\varphi,\mu)} - \frac{2H(\mathcal{E}_{\mathrm{un}}^{(\nu,n)} - v_{\mathrm{eff}}^{(\nu)}(\varphi,\mu))}{\Omega_1^2(\varphi,\mu)}(1 - \frac{H^2}{2\Omega^2(\varphi,\mu)})\Big)d\varphi. \tag{4.13}$$

Suppose that $\varphi_\pm \in [0, 2\pi]$. In this case, inside the interval (φ_-, φ_+) and outside a neighborhood of the turning points φ_- and φ_+, the wave function becomes

$$\psi^{(\nu,n)}(\varphi) = \frac{C^{(n)}}{\big(\mathcal{E}_{\mathrm{un}}^{(\nu,n)} - v_{\mathrm{eff}}^{(\nu)}(\varphi)\big)^{1/4}}\left[\cos\left(\frac{1}{\mu}\int_{\varphi_-}^{\varphi}\sqrt{\mathcal{E}_{\mathrm{un}}^{(\nu,n)} - v_{\mathrm{eff}}^{(\nu)}(\varphi)}\,d\varphi + \theta^{(\nu,n)}(\varphi) + \frac{\pi}{4}\right) + O(\mu)\right], \tag{4.14}$$

where $C^{(n)}$ stands for a normalization constant and

$$\theta_{\mathrm{un}}^{(\nu,n)}(\varphi) = \int_{\varphi_-}^{\varphi}\Big(\frac{3H(\nu_1+1/2)}{\Omega(\varphi,\mu)} - \frac{2H(\mathcal{E}_{\mathrm{un}}^{(\nu,n)} - v_{\mathrm{eff}}^{(\nu)}(\varphi,\mu))}{\Omega_1^2(\varphi,\mu)}(1 - \frac{H^2}{2\Omega^2(\varphi,\mu)}) - \Lambda_{\mathrm{un}}^{(\nu,n)}\Big)d\varphi. \tag{4.15}$$

in a neighborhood of the turning points, one should use another asymptotic representation for $\psi^{(\nu,n)}(\varphi)$, which uses the Airy function or the Maslov canonical operator (see, e.g., [17], [28], and [20]). To obtain $\psi^{(\nu,n)}(\varphi)$ for all values of $\varphi \in \mathbb{R}_\varphi$, one should extend this function to the entire real axis by using formula (4.10). In both the cases, the number n corresponds to the number of oscillations of the function $\psi^{(\nu,n)}(\varphi)$ on the period $[0, 2\pi]$.

The influence of the magnetic field on the eigenvalues is now in the dependence of the frequency ω_0 on H and in the correction with the coefficient μ in formula (3.4). As in the previous case, the effect related to nontrivial quasimomentum (as well as in the long-wave limit) is negligibly small.

Note that condition (4.8) can formally be obtained from the Bohr–Sommerfeld quantization conditions (4.12), (4.13), assuming that $n\mu \ll 1$ and using the Taylor expansion. However, this cannot be done for the asymptotic eigenfunctions [20], [26], [10].

As in Sec. 2 of Part 1, in both the cases 1) and 2), the eigenfunctions are localized in some part of the interval $[0, 2\pi]$ and describe the so-called "trap" states ("trapped" modes) in the waveguide. These states arise only if $\Omega_j(\varphi)$ is nonconstant, whereas $\psi^{(\nu,n)}(\varphi)$ are localized in the semiclassical approximation: these functions are exponentially small outside a neighborhood of the interval $[\varphi_-, \varphi_+]$ (or of the point $\varphi_{\varphi_0^{(\nu)}}$).

4.3. Boundary. The eigenfunctions for the states with the eigenvalues in a neighborhood $V^{(\nu)}_{\max}$ form a boundary layer; to this situation, there correspond the points of the edges $i_{1,2,3}$ of the Reeb graph G placed in a neighborhood of the vertex of branching in the graph. For these states, the Bohr–Sommerfeld rule cannot be applied, and the asymptotic expansion of the eigenfunctions has more complicated structure. Without discussing this case here, we refer the reader to [13], [20], [26], and [34].

4.4. Overbarrier domain and the Aharonov–Bohm phase. In this situation, we have $E^{(\nu,n)} > V^{(\nu)}_{\max}$, and one can use the Bohr–Sommerfeld rule to construct the asymptotic behavior of the eigenvalues. To this situation, there correspond points of edges $i_{2,3}$ of the Reeb graph G that are placed at a distance from the origin. Taking into account the Bloch condition (2.17), we may write

$$E^{(\nu,n)}_{\pm} = \mathcal{E}^{(\nu,n)}_{\rm ov} + \mu\Lambda^{(\nu,n)}_{\rm ov} + O(\mu^2)$$
$$\Lambda^{(\nu,n)}_{\rm ov} = \int_0^{2\pi} \Big(\frac{3H(\nu_1+1/2)}{\Omega(\varphi,\mu)} - \frac{2H(\mathcal{E}^{(\nu,n)}_{\rm ov} - v^{(\nu)}_{\rm eff}(\varphi,\mu))}{\Omega_1^2(\varphi,\mu)}(1 - \frac{H^2}{2\Omega^2(\varphi,\mu)})\Big)d\varphi, \tag{4.16}$$

where $\mathcal{E}^{(\nu,n)}_{\rm ov}$ can be found from the Bohr–Sommerfeld quantization condition,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)}\sqrt{\mathcal{E}^{(\nu,n)}_{\rm ov} - v^{(\nu)}_{\rm eff}(\varphi,\mu)}\,d\varphi = \mu n + \frac{H\rho_0^2}{2\mu}. \tag{4.17}$$

Note that the numbers n take here the values $\sim 1/\mu^2$, whereas the values $\mathcal{E}^{(\nu,n)}$ remain bounded. Since in this case there are no turning points on the entire real axis, we see that the following global representation holds for the eigenfunctions:

$$\psi^{(\nu,n)}_{\pm}(\varphi) = \frac{C^{(n)}_{\pm}}{\big(\mathcal{E}^{(\nu,n)}_{\rm un} - v^{(\nu)}_{\rm eff}(\varphi)\big)^{1/4}}\sqrt{\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}}\times$$
$$\Big[\exp\Big(\frac{\pm i}{\mu}\int_{\varphi_0}^{\varphi}\frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)}\sqrt{\mathcal{E}^{(\nu,n)}_{\rm un} - v^{(\nu)}_{\rm eff}(\varphi)}\,d\varphi + \theta^{(\nu,n)}(\varphi)\Big) + O(\mu)\Big], \tag{4.18}$$

Here φ_0 is a chosen point and $C^{(n)}_{\pm}$ stands for the normalization constant. There are no focal points above the barrier, and the "ballistic transport" takes place (see the right part of Fig. 4). In this case, the influence of the magnetic field is visible because the magnetic flow enters the quantization conditions (via the Aharonov–Bohm phase). In contrast to the example in Sec. 2 of part 2, the multiplicity of the spectrum depends on the magnetic flow. For example, if the flow is not an integer, then the eigenvalue $E^{(\nu,n)}_{\pm}$ is nondegenerate. However, if the flow is an integer, at least asymptotically, then the spectrum is degenerate, and two eigenfunctions correspond to the eigenvalue $E^{(\nu,n)}_{\pm}$. In turn, this means that the so-called "Zitterbewegung" occurs for these magnetic fields.

5. ULTRA-SHORT REGIMES AND THE BORN APPROXIMATION

5.1. *Formulas for the Born approximation*

The investigation similar to that carried out in Sec. 4 of part I in [10] shows that the formulas of the previous paragraph work well provided that not only $\mathcal{E}^{\nu_1\nu_2 n} \gg \max(v_{\text{eff}}^{\nu_1\nu_2}(\varphi))$ but also $1/\mu \gg \mathcal{E}^{\nu_1\nu_2 n}$. Here we can simplify the expressions (4.17) and (4.18) by expending the radicand (in the Taylor series)

$$\sqrt{\mathcal{E} - \frac{1}{2} v_{\text{eff}}^{(\nu)}}(\varphi) = \sqrt{\mathcal{E}} - \frac{v_{\text{eff}}^{(\nu)}(\varphi)}{\sqrt{\mathcal{E}}} + O(\frac{1}{(\mathcal{E})^{3/2}}).$$

Up to summands of of higher order of smallness, this gives

$$E_{\pm}^{(\nu,n)} \approx \mathcal{E}_{\text{born}}^{(\nu,n)} + \mu\Lambda_{\text{born}}^{(\nu,n)}, \tag{5.1}$$

where

$$\mathcal{E}_{\text{ov}}^{(\nu,n)} = \big(2\pi(\mu n + \frac{H\rho_0^2}{2\mu})\big)^2 \Big/ \Big(\int_0^{2\pi} \frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)} d\varphi\Big)^2 \tag{5.2}$$

$$\Lambda_{\text{born}}^{(\nu,n)} = \int_0^{2\pi} \Big(\frac{3H(\nu_1 + 1/2)}{\Omega(\varphi,\mu)} - \frac{2H(\mathcal{E}_{\text{ov}}^{(\nu,n)} - v_{\text{eff}}^{(\nu)}(\varphi,\mu))}{\Omega_1^2(\varphi,\mu)}(1 - \frac{H^2}{2\Omega^2(\varphi,\mu)})\Big) d\varphi,$$

For the wave function, we obtain

$$\Psi^{(\nu,n)} \approx C_{\pm}^{(n)} \chi_0^{\nu}\Big(\frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)}\sqrt{\mathcal{E}_{\text{ov}}^{(\nu,n)}}, \phi, y, z\Big)\sqrt{\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}}$$
$$\times \exp\Big(\pm(i/\mu)\sqrt{\mathcal{E}_{\text{un}}^{(\nu,n)}}\int_{\varphi_0}^{\varphi}\frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)}d\varphi + \theta_{\text{born}}^{(\nu,n)}(\varphi)\Big),$$

$$\theta_{\text{ov}}^{(\nu,n)}(\varphi) = \int_{\varphi_0}^{\varphi}\Big(-\frac{\Omega(\varphi,\mu)v_{\text{eff}}^{(\nu)}(\varphi)}{\mu\sqrt{\mathcal{E}_{\text{born}}^{(\nu,n)}}\Omega_1(\varphi,\mu)} + \frac{3H(\nu_1+1/2)}{\Omega(\varphi,\mu)}$$
$$- \frac{2H(\mathcal{E}_{\text{ov}}^{(\nu,n)} - v_{\text{eff}}^{(\nu)}(\varphi,\mu))}{\Omega_1^2(\varphi,\mu)}(1 - \frac{H^2}{2\Omega^2(\varphi,\mu)}) - \Lambda_{\text{born}}^{(\nu,n)}\Big)d\varphi$$

The formulas thus obtained coincide in essence with the formulas of the so-called Born approximation (see, for example, [23]); in this case, the leading parts of the phase and of the expansion of energy are determined by the kinetic energy

$$\frac{p_\varphi^2}{2\rho_0^2}\left(\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}\right)^2$$

of the effective (classical) Hamiltonian. The effective potential, as well as its contribution to the spectrum, can be taken into account in the form of corrections.

5.2 *Correction $\chi_1^{(\nu)}$ and the destruction of the local description*

The dependence of $\chi(\nu)_1$ on p_ϕ is a restriction for applying the formulas of Subsec. 5.2. A rather cumbersome investigation of the formula for $\chi(\nu)_1$ shows that $\chi_1^{(\nu)}$ can be represented as a sum of the functions

$$\sum_{|\sigma-\nu|\leqslant 4} A^{(\nu,\sigma)}(\varphi, p_\varphi)\chi_0^{(\sigma)}, \tag{5.3}$$

where $A^{\nu,\sigma}(\varphi, p_\varphi)$ are polynomials in p_φ of degree not exceeding two with 2π-periodic coefficients smoothly depending on φ. For the energy values $E \sim \mu^{-\gamma}$, the momentum p_φ takes the value $\sim \mu^{-\gamma/2}$, and the correction generated by the operator $\mu\hat{\chi}(\nu)_1$ takes the value $\mu^{1-\gamma}$. This correction is small if $1 > \gamma$, and the corresponding correction increases as γ approaches 1.

5.3. *Displacement of the geometric axis for a strongly perturbed motion*

In the approximation under consideration, one can represent the exponential function

$$\exp\Big(\pm(i/\mu)\sqrt{\mathcal{E}_{\mathrm{un}}^{(\nu,n)}}\int_{\varphi_0}^{\varphi}\frac{\Omega(\varphi,\mu)}{\Omega_1(\varphi,\mu)}d\varphi+\theta_{\mathrm{born}}^{(\nu,n)}(\varphi)\Big)$$

(entering the definition of the function $\Psi^{(\nu,n)}$) in the form $\exp\frac{iS(\varphi)}{h}$, $h=\mu^{1+\gamma/2}$. We thus have a semiclassical asymptotic expansion with the parameter h. To extend the domain of applicability of the asymptotic formulas, one can try to apply the asymptotic expansions with the parameter h already at the original ansatz. The related investigation shows that, to this end, one should "displace" the axis of the waveguide by replacing the value y_0 in the corresponding formulas by

$$y_1=y_0\left(1-\frac{\mu r_0}{\rho_0}\right),$$

and the effective Hamiltonian should also be corrected by including summands of "high dispersion,"

$$H_{\mathrm{eff}}^{(\nu_1,\nu_2)}(p_\varphi,\varphi)=\frac{p_\varphi^2}{2\Omega(\varphi,\mu)^2\rho_0^2}\Big[\Omega_1(\varphi,\mu)^2+2H\mu p_\varphi\rho_0^{-2}-\big(\mu p_\varphi\rho_0^{-2}\big)^2\Big]+\Omega(\varphi,\mu)(\nu_1+1/2)$$
$$+\Omega_2(\varphi,\mu)(\nu_2+1/2).$$

This modification of the adiabatic approximation enables us to eliminate the powers p_φ^2 from the coefficients A^σ in (5.3) and "extend" the domain of applicability of this approximation to the values of the parameter γ of the form $\gamma=2-\varepsilon$, $\varepsilon>0$. It is impossible to eliminate the powers p_φ from A^σ, which gives a "destruction" of the adiabatic approximation for the values $\gamma\geqslant 2$. The physical explanation of this fact is just like that presented in part I of [10] for a straight-line waveguide. One cannot apply a local description for the case in which the order of time of the particle flyby along the entire waveguide is the order of the period of transversal oscillations of the particle.

6. SUPEREXCITED STATES

Let us study the situation for which $\gamma=2$ in more detail by using the complex germ theory, see [29] and [3] (and also [1], [14], [31], and [35]), modified in the spirit of [22] and [15] for the case of individual invariant closed trajectories. As in Sec. 5 of part 1, in this case, it is convenient to represent equation (1.5) (with the Hamiltonian (1.6)) in the form

$$\frac{1}{2}\Bigg[-h^2\frac{\partial^2}{\partial\eta^2}+\frac{1}{(\rho_0+\eta)^2}\left(-ih\frac{\partial}{\partial\varphi}+H\rho_0\eta+\frac{H\eta^2}{2}\right)^2-\frac{h^2}{4(\rho_0+\eta)^2}-h^2\frac{\partial^2}{\partial\zeta^2}$$
$$+\Omega_1^2(\varphi,\mu)\eta^2+\Omega_2^2(\varphi,\mu)\zeta^2\Big]\Psi=hE\Psi, \tag{6.1}$$

where $h=\mu^2$, $\eta=\mu y,\zeta=\mu z$. The symbol of the "h"-differential operator corresponding to the Hamiltonian on the left-hand side of equation (6.1) is

$$\mathcal{H}(p_\eta,\eta,p_\zeta,\zeta,p_\varphi,\varphi,h)=\mathcal{H}_0(p_\eta,\eta,p_\zeta,\zeta,p_\varphi,\varphi)+h^2\mathcal{H}_2(p_\eta,\eta,p_\zeta,\zeta,p_\varphi,\varphi),$$
$$\mathcal{H}_0=\frac{p_\eta^2}{2}+\frac{1}{2(\rho_0+\eta)^2}\left(p_\varphi+H\rho_0\eta+\frac{H\eta^2}{2}\right)^2+\frac{p_\zeta^2}{2}+\frac{\Omega_1^2(\varphi,\mu)\eta^2}{2}+\frac{\Omega_2^2(\varphi,\mu)\zeta^2}{2},$$
$$\mathcal{H}_2=-\frac{h^2}{4(\rho_0+\eta)^2}.$$

The Hamiltonian system corresponding to the classical Hamiltonian $\mathcal{H}_0$ is

$$\dot p_\eta = \frac{1}{(\rho_0+\eta)^3}\left(p_\varphi + H\rho_0\eta + \frac{H\eta^2}{2}\right)\left(H\rho_0^2 + H\rho_0\eta + \frac{H\eta^2}{2} - p_\varphi\right) - \Omega_1^2(\varphi,\mu)\eta, \quad \dot\eta = p_\eta,$$
$$\dot p_\zeta = -\Omega_2^2(\varphi,\mu)\zeta, \quad \dot\zeta = p_\zeta,$$
$$\dot p_\varphi = -\Omega_1(\varphi,\mu)\Omega_1'(\varphi,\mu)\eta^2 - \Omega_2(\varphi,\mu)\Omega_2'(\varphi,\mu)\zeta^2, \quad \dot\varphi = \frac{1}{(\rho_0+\eta)^2}\left(p_\varphi + H\rho_0\eta + \frac{H\eta^2}{2}\right). \tag{6.2}$$

It can readily be seen that this system admits a closed classical trajectory Γ lying on the energy level $E = \mathcal{H}_0|_\Gamma = H^2\rho_0^2/2$ with the frequency $\omega = H$; this trajectory is defined by the formula

$$\Gamma = \{p_\eta = 0, \quad \eta = 0, \quad p_\zeta = 0, \quad \zeta = 0, \quad p_\varphi = \text{ const} = H\rho_0^2, \quad \varphi = Ht + \varphi_0\}.$$

Further, according to [29], one should study the so-called "system in variations." This system consists of six equations; however, we need only four of them, namely, the equations containing η and ζ. Denote the corresponding components by $\delta p_\eta = W_1$, $\delta\eta = Z_1$, $\delta p_\zeta = W_2$, and $\delta\zeta = Z_2$. Note that one can set $\eta = 0$ in the first two factors entering the first term in (6.2) and differentiate the third factor only. We then obtain

$$\dot W_1 = -\left[H^2 + \Omega_1^2(Ht+\varphi_0)\right] Z_1, \quad \dot Z_1 = W_1, \quad \dot W_2 = -\Omega_2^2(Ht+\varphi_0)Z_2, \qquad \dot Z_2 = W_2. \tag{6.3}$$

These equations reduce to the Hill equations

$$\ddot Z_1 + \left[H^2 + \Omega_1^2(Ht+\varphi_0)\right] Z_1 = 0, \quad \ddot Z_2 + \Omega_2^2(Ht+\varphi_0)Z_2 = 0. \tag{6.4}$$

To the trajectory Γ there corresponds a spectral series, i.e., a subsequence of the eigenvalues of the operator in (6.1), if this trajectory is orbitally stable in the linear approximation or, in other words, if equations (6.4) admit stable Floquet solutions Z_1 and Z_2. In this subsection we assume that this condition holds (this depends, in particular, on the magnetic intensity H of the field). Denote the corresponding Floquet exponents by β_1 and β_2; then the desired solutions of the Floquet system (6.3) can be represented in the form

$$Z_1 = Z_1^{(0)}(\varphi)\exp(i\beta_1 t), \qquad W_1 = W_1^{(0)}(\varphi)\exp(i\beta_1 t),$$
$$Z_2 = Z_2^{(0)}(\varphi)\exp(i\beta_2 t), \qquad W_2 = W_2^{(0)}(\varphi)\exp(i\beta_2 t),$$
$$\varphi = Ht+\varphi_0, \quad \bar W_1^{(0)} Z_1^{(0)} - W_1^{(0)}\bar Z_1^{(0)} = 2i, \quad \bar W_2^{(0)} Z_2^{(0)} - W_2^{(0)}\bar Z_2^{(0)} = 2i,$$

where $\varphi = Ht+\varphi_0$ and the functions $W_1^{(0)}$, $Z_1^{(0)}$, $W_2^{(0)}$, and $Z_2^{(0)}$ are 2π-periodic with respect to φ and normalized by the conditions $\bar W_1^{(0)} Z_1^{(0)} - W_1^{(0)}\bar Z_1^{(0)} = 2i$ and $\bar W_2^{(0)} Z_2^{(0)} - W_2^{(0)}\bar Z_2^{(0)} = 2i$. Note that, under the above assumption on the stability of Γ, as follows from [4], the Hamiltonian $\mathcal{H}_0$ can be represented in the form

$$\begin{aligned}\mathcal{H}_0 &= \mathcal{H}_0|_\Gamma + \omega(I_0 - H\rho_0^2) + \beta_1 I_1 + \beta_2 I_2 + O((I_0 - H\rho_0^2)^2 + I_1^2 + I_2^2)\\ &= H^2\rho_0^2/2 + H(I_0 - H\rho_0^2) + \beta_1 I_1 + \beta_2 I_2 + O((I_0 - H\rho_0^2)^2 + I_1^2 + I_2^2)\end{aligned} \tag{6.5}$$

in a neighborhood of the curve Γ. Here I_0, I_1, and I_2 are variables of action type which "approximately" characterize the classical trajectories of the Hamiltonian system (6.2) in a neighborhood of the trajectory Γ. These variables are quantized in the subsequent constructions, and this very quantization gives a spectral series (a part of the spectrum) corresponding to a neighborhood of the trajectory Γ (see also [22] and [15]).

Using formulas [29], we can construct the wave function of equation (1.3). In the original variables, with regard to the additional oscillating factor

$$\exp\bigl(-iH\rho_0^2\varphi/(2\mu^2)\bigr),$$

this wave function becomes

$$\Psi = \frac{e^{i\theta(\varphi)}}{\sqrt{Z_1^{(0)}(\varphi)Z_2^{(0)}(\varphi)}}\exp\left(\frac{iH\rho_0^2\varphi}{2\mu^2} + \frac{i}{2}\frac{\bar W_1^{(0)}}{Z_1^{(0)}}(\varphi)y^2 + \frac{i}{2}\frac{\bar W_2^{(0)}}{Z_2^{(0)}}(\varphi)z^2\right)H_{\nu_1}(q_1 y)H_{\nu_2}(q_2 z), \qquad (6.6)$$

where

$$\theta(\varphi) = -\beta_1(\nu_1 + 1/2)\varphi - \beta_2(\nu_1 + 1/2)\varphi + \lambda\varphi + \theta_0(\varphi)$$

for

$$\theta_0(\varphi) = (\nu_1 + 1/2)\arctan\frac{\bar W_1^{(0)}}{Z_1^{(0)}} + (\nu_2 + 1/2)\arctan\frac{\bar W_2^{(0)}}{Z_2^{(0)}}, \quad q_1 = \sqrt{\left|\frac{\bar W_1^{(0)}}{Z_1^{(0)}}\right|}, \quad q_2 = \sqrt{\left|\frac{\bar W_2^{(0)}}{Z_2^{(0)}}\right|}.$$

Here λ stands for a "free" parameter independent of μ, and the energy E takes the values

$$E = \frac{H^2\rho_0^2}{2\mu^2} + H\rho_0^2\lambda + O(\mu^2)$$

For the function in (6.6) to be single-valued, it is necessary that

$$\lambda = \lambda^{(\nu,n)} = H\rho_0^2/(2\mu^2) + \beta_1(\nu_1 + 1/2) + \beta_2(\nu_1 + 1/2) - n + O(\mu^2).$$

Finally, for the energy levels E of the original operator (1.2), we obtain the formula

$$E = E^{(\nu,n)} = H^2\rho_0^2/2 + H(\mu n - H\rho_0^2) + \mu\beta_1(1/2 + \nu_1) + \mu\beta_2(1/2 + \nu_1) + O(\mu^2). \qquad (6.7)$$

Here n stands for (large) integers belonging to a neighborhood of the integer part of the number $H\rho_0^2/(2\mu^2)$. This gives a discrete family of eigenvalues (a "spectral series") of the quantum waveguide with large energy values which correspond in the classical limit to the trajectory Γ. Note that the "power" of this spectral series is substantially less than the "powers" of the spectral series constructed in the previous sections. Up to a correction of the order of $O(\mu^2)$, these values are obtained from formula (6.5) as a result of the quantization of the variables I_0, I_1, and I_2 given by

$$I_0 = \mu n, \qquad I_1 = \mu(\frac{1}{2} + \nu_1), \qquad I_2 = \mu(\frac{1}{2} + \nu_1).$$

It is clear from formula (6.6) that, in contrast to the situation in which the adiabatic approximation works well, the influence of soft walls given by the coefficients $\Omega_1(\varphi,\mu)$ and $\Omega_2(\varphi,\mu)$ is of "integral" nature, which defines the dependence of the wave function, in the direction orthogonal to the axis of the waveguide, via the solutions of the "system in variations" Z_1, Z_2 characterizing the total motion of the particle along the waveguide. In a sense, this phenomenon can be assumed to be related to the so-called Fermi acceleration [36].

APPENDIX A

We present here the evaluation of the subprincipal symbol $\mathcal{L}_1^{(\nu)}(p_\varphi, \varphi)$. For the function $\chi_1^{(\nu)}$, we have the equation

$$(\mathcal{H}_0 - H_{\mathrm{eff}}^{(\nu)})\chi_1^{(\nu)} = \frac{\partial \mathcal{H}_0}{\partial p_\varphi}\frac{\partial \chi_0^{(\nu)}}{\partial \varphi} - \frac{\partial H_{\mathrm{eff}}^{(\nu)}}{\partial \varphi}\frac{\partial \chi_0^{(\nu)}}{\partial p_\varphi} - \mathcal{H}_1\chi_0^{(\nu)} + \chi_0^{(\nu)}\mathcal{L}_1^{(\nu)},$$

and also the relations

$$\frac{\partial \mathcal{H}_0}{\partial p_\varphi} = \frac{p_\varphi}{\rho_0^2}\left(\frac{\Omega_1(\varphi,\mu)}{\Omega(\varphi,\mu)}\right)^2 + \frac{H(y+y_0)}{\rho_0}, \qquad \frac{\partial \chi_0^{(\nu)}}{\partial p_\varphi} = \frac{H}{\rho_0\Omega^2(\varphi,\mu)}\frac{\partial \chi_0^{(\nu)}}{\partial y},$$
$$\frac{\partial \chi_0^{(\nu)}}{\partial \varphi} = 1/2\frac{\partial}{\partial \varphi}\left[\log \Omega(\varphi,\mu)\right]\left(\frac{\chi_0}{2} + y\frac{\partial \chi_0^{(\nu)}}{\partial y}\right) + 1/2\frac{\partial}{\partial \varphi}\left[\log \Omega_2(\varphi,\mu)\right]\left(\frac{\chi_0}{2} + z\frac{\partial \chi_0^{(\nu)}}{\partial z}\right).$$

The condition that the right-hand side of this equation is orthogonal to the kernel of the self-adjoint operator on the left-hand side gives the following relation for $\mathcal{L}_1^{(\nu)}$ (see [6]):

$$\mathcal{L}_1^{(\nu)} = \langle \chi_0^{\nu_1\nu_2}, \mathcal{H}_1\chi_0^{\nu_1\nu_2}\rangle_{yz} - i\left\langle \chi_0^{\nu_1\nu_2}, \frac{d\chi_0^{\nu_1\nu_2}}{dt}\right\rangle_{yz} - i\left\langle \chi_0^{\nu_1\nu_2}, \left[\frac{\partial \mathcal{H}_0}{\partial p_\varphi} - \frac{\partial H_{\mathrm{eff}}}{\partial p_\varphi}\right]\frac{\partial \chi_0^{\nu_1\nu_2}}{\partial \varphi}\right\rangle_{yz}, \tag{6.8}$$

where the angular brackets $\langle\cdot,\cdot\rangle_{yz}$ stand for the inner product in the space $L_2(\mathbb{R}^2_{yz})$, and the operator d/dt is defined as follows:

$$\frac{d}{dt} = \frac{\partial H_{\mathrm{eff}}^{\nu_1\nu_2}}{\partial p_\varphi}\frac{\partial}{\partial \varphi} - \frac{\partial H_{\mathrm{eff}}^{\nu_1\nu_2}}{\partial \varphi}\frac{\partial}{\partial p_\varphi}.$$

The last two summands in (6.8) together give "Berry's phase"[1]. We have $\langle \chi_0^{(\nu)}, d\chi_0^{(\nu)}/dt\rangle_{yz} = 0$ for $\chi_0^{(\nu)}$, since $\chi_0^{(\nu)}$ is a real-valued function normalized by one. The evaluation of the third summand gives

$$-i\left\langle \chi_0, \left[\frac{\partial \mathcal{H}_0}{\partial p_\varphi} - \frac{\partial H_{\mathrm{eff}}}{\partial p_\varphi}\right]\frac{\partial \chi_0}{\partial \varphi}\right\rangle_{yz} = -\frac{iH}{\rho_0}\left\langle \chi_0, (y+y_0)\frac{\partial \chi_0}{\partial \varphi}\right\rangle_{yz}$$
$$= -i\frac{\partial}{\partial \varphi}\left(\frac{H^2}{\Omega^2(\varphi,\mu)}\right)\frac{p_\varphi}{\rho_0^2}\int_{-\infty}^{\infty}\zeta^{\nu_1}(\xi)\xi\frac{\partial \zeta^{\nu_1}}{\partial \xi}(\xi)\,d\xi = -i\frac{\partial}{\partial \varphi}\left(\frac{\Omega_1^2(\varphi,\mu)}{\Omega^2(\varphi,\mu)}\right)\frac{p_\varphi}{2\rho_0^2}, \tag{6.9}$$

where we have used the notation $\xi = \sqrt{\Omega(\varphi,\mu)}(y+y_0)$. Therefore, the symbol $\mathcal{L}_1^{(\nu)}$ is

$$\mathcal{L}_1^{(\nu)} = \frac{p_\varphi^2}{\rho_0^3}y_0 - \frac{3Hp_\varphi}{2\rho_0^2}\left(y_0^2 + \frac{Q_{\nu_1}}{\Omega^2}\right) + \frac{H^2}{2\rho_0}\left(y_0^3 + \frac{3y_0Q_{\nu_1}}{\Omega^2}\right) - i\frac{\partial}{\partial \varphi}\left(\frac{\Omega_1^2(\varphi,\mu)}{\Omega^2(\varphi,\mu)}\right)\frac{p_\varphi}{2\rho_0^2} =$$
$$= \frac{\Omega_1^2(\varphi,\mu)}{\Omega^4(\varphi,\mu)}\left(1 - \frac{H^2}{2\Omega^2}\right)\frac{Hp_\varphi^3}{\rho_0^4} - \frac{3HQ_{\nu_1}\Omega_1^2(\varphi,\mu)}{2\rho_0^2\Omega^4(\varphi,\mu)}p_\varphi - i\frac{\partial}{\partial \varphi}\left(\frac{\Omega_1^2(\varphi,\mu)}{\Omega^2(\varphi,\mu)}\right)\frac{p_\varphi}{2\rho_0^2}. \tag{6.10}$$

[1]M. V. Berry, "Quantal phase factors accompanying adiabatic changes," Proc. Roy. Soc. London Ser. A **392** (1802), 45–57 (1984).

APPENDIX B

To evaluate the symbol $\mathcal{L}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}$, we use the orthogonality conditions for the right-hand side of the second approximation (see (3.15) in [6]). We obtain $\mathcal{L}_2^{(\nu)}=-(\chi_0^{(\nu)},F_2)+(\chi_0^{(\nu)},\mathcal{H}_2\chi_0^{(\nu)})$ and

$$F_2=i\left[\frac{\partial\mathcal{H}_0}{\partial p_\varphi}\frac{\partial\chi_1^{(\nu)}}{\partial\varphi}-\frac{\partial H_{\rm eff}^{(\nu)}}{\partial\varphi}\frac{\partial\chi_1^{(\nu)}}{\partial p_\varphi}\right]-\mathcal{H}_1\chi_1^{(\nu)}+\chi_1^{(\nu)}\mathcal{L}_1^{(\nu)}$$
$$+i\left[\frac{\partial\mathcal{H}_1}{\partial p_\varphi}\frac{\partial\chi_0^{(\nu)}}{\partial\varphi}-\frac{\partial\mathcal{L}_1^{(\nu)}}{\partial\varphi}\frac{\partial\chi_0^{(\nu)}}{\partial p_\varphi}\right]+\frac{1}{2}\sum_{i,j}\left[\frac{\partial^2\mathcal{H}_0}{\partial p_\varphi^2}\frac{\partial^2\chi_0^{(\nu)}}{\partial\varphi^2}-\frac{\partial^2 H_{\rm eff}^{(\nu)}}{\partial\varphi^2}\frac{\partial^2\chi_0^{(\nu)}}{\partial p_\varphi^2}\right].$$

It is clear that, when evaluating $\mathcal{L}_2^{(\nu)}(p_\varphi,\varphi)|_{p_\varphi=0}$, one should set $p_\varphi=0$ in the definition of F_2. We also take into account that, up to $o(\mu^2)$ $\chi_0^{(\nu)}$, the Hamiltonians $H_{\rm eff}^{(\nu)}$ and $\mathcal{H}_0$ do not depend on φ, together with the relations $\frac{\partial\mathcal{H}_0}{\partial p_\varphi}|_{p_\varphi=0}=0$, $\mathcal{L}_1^{(\nu)}|_{p_\varphi=0}=0$, and $(\chi_0^{(\nu)},\chi_1^{(\nu)})=0$. This gives

$$\mathcal{L}_2^{(\nu)}|_{p_\varphi=0}=(\mathcal{H}_1|_{p_\varphi=0}\chi_0^{(\nu)},\chi_1^{(\nu)})+(\mathcal{H}|_{2p_\varphi=0}\chi_0^{(\nu)},\chi_0^{(\nu)})$$
$$=-\frac{H^2}{2\rho_0}(y^3\tilde\chi_0^{(\nu)},\tilde\chi_1^{(\nu)})+\frac{H^2}{2\rho_0^2}(y^2\tilde\chi_0^{(\nu)},y^2\tilde\chi_0^{(\nu)})-\frac{1}{8\rho_0^2}(\tilde\chi_0^{(\nu)},\tilde\chi_0^{(\nu)}),$$

where $\tilde\chi_j^{(\nu)}=\chi_j^{(\nu)}|_{p_\varphi=0}$. Let us use the known relations

$$y^2\tilde\chi_0^{(\nu)}=\sum_{k=-2,k\neq0}^{k=2}a_k^{(\nu)}\tilde\chi_0^{(\nu_1+k,\nu_2)},\quad y^3\tilde\chi_0^{(\nu)}=\sum_{k=-3,k\neq0}^{k=3}b_k^{(\nu)}\tilde\chi_0^{(\nu_1+k,\nu_2)},\tag{6.11}$$

where

$$a_{-2}^{(\nu)}=\frac{1}{2\Omega^{(0)}}\sqrt{\nu_1(\nu_1-1)},\ a_{-1}^{(\nu)}=0,\ a_0^{(\nu)}=\frac{1}{2\Omega^{(0)}}(2\nu_1+1)$$
$$a_1^{(\nu)}=0,\ a_2^{(\nu)}=\frac{1}{2\Omega^{(0)}}\sqrt{(\nu_1+1)(\nu_1+2)},$$
$$b_{-3}^{(\nu)}=\frac{\sqrt2}{4(\Omega^{(0)})^2}\sqrt{\nu_1(\nu_1-1)(\nu_1-2)},\ b_{-2}^{(\nu)}=0,\ b_{-1}^{(\nu)}=\frac{3\sqrt2}{4(\Omega^{(0)})^2}\nu_1^{3/2},\ b_0^{(\nu)}=0,$$
$$b_1^{(\nu)}=\frac{3\sqrt2}{4(\Omega^{(0)})^2}(\nu_1+1)^{3/2},\ b_2^{(\nu)}=0,\ b_3^{(\nu)}=\frac{\sqrt2}{4(\Omega^{(0)})^2}\sqrt{(\nu_1+1)(\nu_1+2)(\nu_1+3)}.$$

The function $\chi_1^{(\nu)}$ can be found from equation (6.13), which, for $p_\varphi=0$ and for the function $\tilde\chi_1^{(\nu)}$, becomes

$$\frac12\left[-\frac{\partial^2}{\partial y^2}+{\Omega^{(0)}}^2y^2-\frac{\partial^2}{\partial z^2}+{\Omega_2^{(0)}}^2z^2\right]\tilde\chi_1^{(\nu)}-\lambda_\nu\tilde\chi_1^{(\nu)}=\frac{H^2}{2\rho_0}y^3\tilde\chi_0^{(\nu)},\tag{6.12}$$

where $\lambda_\nu=\Omega^{(0)}(\nu_1+1/2)+\Omega_2^{(0)}(\nu_2+1/2)$. Using the second relation in (6.11) and applying the Fourier method, we find the expansion

$$\tilde\chi_1^{(\nu)}=\frac{1}{\Omega^{(0)}}\sum_{k=-3,k\neq0}^{k=3}\frac{b_k^{(\nu)}}{k}\tilde\chi_0^{(\nu_1+k,\nu_2)}.\tag{6.13}$$

This yields

$$\mathcal{L}_2^{(\nu)}|_{p_\varphi=0} = -\frac{H^2}{2\rho_0\Omega^{(0)}} \sum_{k=-3,k\neq 0}^{k=3} \frac{|b_k^{(\nu)}|^2}{k} + \frac{H^2}{2\rho_0^2} \sum_{k=-2,k\neq 0}^{k=2} |a_k^{(\nu)}|^2 - \frac{1}{8\rho_0^2}$$
$$= -\frac{H^2}{16\rho_0(\Omega^{(0)})^5}(12\nu_1^2 + 12\nu_1 + 5) + \frac{3H^2}{8\rho_0^2(\Omega^{(0)})^2}(2\nu_1^2 + 2\nu_1 + 1) - \frac{1}{8\rho_0^2},$$

ACKNOWLEDGMENTS

This research was supported by the DFG–RAS project, the grant no. 2.1.1/4540 of the Ministry of Science and Education of the Russian Federation, and by the Program of the Presidium of the RAS no. 23.

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