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On the Nodal Count for Flat Tori*

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Abstract: We give an explicit formula for the number of nodal domains of certain eigenfunctions on a flat torus. We apply this to an isospectral but not isometric family of pairs of flat four-dimensional tori constructed by Conway and Sloane, and we show that corresponding eigenfunctions have the same number of nodal domains. This disproves a conjecture by Brüning, Gnutzmann, and Smilansky.

Introduction

The discovery and measurement of the dark lines in the solar spectrum by Fraunhofer and the fundamental idea of spectral analysis, as conceived and corroborated by Bunsen and Kirchhoff, already showed that one has to approach the physics of the very large (and the very small) mainly via the solution of inverse spectral problems. Von Neumann's Spectral Theorem lead to a mathematical formulation of this task, by associating to any self-adjoint operator, Δ, an increasing family of orthogonal projections, $\big(E_\Delta(\lambda)\big)_{\lambda\in\mathrm{spec}\,\Delta}$, which describes the operator completely; in the discrete case, $E_\Delta(\lambda)$ is given by the orthogonal sum of the eigenspaces of Δ with eigenvalue at most λ (here and below, we denote subspaces of a Hilbert space and their orthogonal projections by the same letter). Thus, solving the inverse spectral problem means to derive the characteristics of a physical system from the spectrum of its Hamiltonian Δ. If Δ is discrete and semibounded, then the eigenvalue counting function

$$N(\lambda) := \mathrm{tr}\, E_\Delta(\lambda) \tag{0.1}$$

determines a complete set of unitary invariants for Δ in the given Hilbert space. This situation may change, however, if we restrict the class of admissible unitary transformations. In a very influential paper [K], Kac proposed to study the inverse spectral problem

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for bounded vibrating membranes $M \subset \mathbb{R}^m$ which may be thought of as "drums". For those, Weyl's celebrated eigenvalue asymptotics,

$$N_\Delta(\lambda) \sim_{\lambda\to\infty} C_m \operatorname{vol} M \lambda^{m/2}, \tag{0.2}$$

suggested that they could be determined by the spectrum up to a Euclidean isometry, and Kac elaborated on that. However, in the case of closed membranes–actually flat tori–Milnor [M] had already published a counterexample, such that the new field of *spectral geometry*, quickly emerging in reaction to Kac' impulse, undertook to analyze the precise relationship between the geometry of a compact Riemannian manifold M with (reasonable) boundary and the spectrum of a self-adjoint elliptic boundary value problem for the scalar Laplacian Δ_M. Many generalizations and ramifications have evolved in the course of time; for some recent survey articles see [Z] and [GPS].

The results of this endeavour so far can be roughly split into "positive" and "negative" ones, those identifying metric invariants–like dimension or volume–which are determined by the spectrum, and those showing for other such quantities that they are not spectrally determined–like the homeomorphism type, even of spheres or balls, through the construction of specific isospectral pairs. By and large, the negative results outweigh the positive ones by far such that it is natural to look for additional information beyond the spectrum which resolves isospectrality into isometry. This requires more information from the spectral decomposition of Δ_M. For example, the Dirichlet spectrum of the Laplacian on a compact manifold M with smooth nonempty boundary ∂M will not suffice to determine the isometry class of M but this is the case if we know in addition the normal derivatives of an orthonormal basis, $(\psi_j)_{j=1}^\infty$, of Dirichlet eigenfunctions along ∂M; see the monograph [KKL] where also reconstruction algorithms are given.

In the case of real operators like the Laplacian, we may restrict attention to real eigenfunctions. While they are, in general, very complicated and explicitly known only in a few special cases, the zero set–or set of *nodal points* $\psi^{-1}(0)$, displays attractive features. This was discovered by Ernst Florens Friedrich Chladni (1756–1827) who baffled the intellectual elite of his day with his "Klangfiguren" or "visual acoustics", which he produced as sand patterns on a glass plate by exciting the plate with a violin bow. The nodal pattern of eigenfunctions has been studied in Mathematical Physics ever since, with many interesting applications, cf. the survey [SmSt].

Probably the simplest aspect of the nodal set of a real nonzero eigenfunction, ψ, is its *nodal count*,

$$\operatorname{nc}(\psi) := \text{ number of connected components of } \psi^{-1}(\mathbb{R}\backslash\{0\}); \tag{0.3}$$

for the zero eigenfunction the nodal count is zero. Any connected component of $\psi^{-1}(\mathbb{R}\backslash\{0\})$ will be called a *nodal domain* of ψ, and a *positive* or *negative* nodal domain according to whether ψ is positive or negative there.

One of the few general results on the nodal pattern, Courant's Theorem [CH, p. 393] together with the Weyl asymptotics (0.2) implies that there is a bound on the nodal count in terms of the eigenvalue,

$$\operatorname{nc}(\psi) \le C\lambda^{2/m}, \quad m := \dim M, \tag{0.4}$$

for $\psi \in E_\Delta(\lambda)$ and some constant $C = C(M)$. Since the number of nodal domains does not change under multiplication by a nonzero constant, we can introduce the *nodal count*, nc_Δ, of Δ as the function

$$\operatorname{nc}_\Delta(\lambda) := \{\operatorname{nc}(\psi) : \psi \in \mathbb{P}E_\Delta(\lambda)\} \subset \mathbb{Z}_+, \tag{0.5}$$

which takes values in the finite subsets of $\mathbb{Z}_+$. In the case of two isospectral discrete Laplacians, Δ_1 and Δ_2, we will say that they have *the same nodal count* or that they are *isonodal*, if there is a bijection,

$$\Psi(\lambda) : \mathbb{P}E_{\Delta_1}(\lambda) \to \mathbb{P}E_{\Delta_2}(\lambda) \tag{0.6}$$

such that

$$\mathrm{nc}(\Psi_\lambda(\psi)) = \mathrm{nc}(\psi), \tag{0.7}$$

for all $\psi \in \mathbb{P}E_{\Delta_1}(\lambda)$.

On the basis of some experimental evidence, it has been conjectured recently by J. Brüning, S. Gnutzmann, and U. Smilansky that the nodal count resolves isospectrality. This conjecture has been corroborated since then in a number of cases, cf. [BSS,GSS, BKP] (note that in [GSS,BKP] a different definition of nodal count has been employed). For certain restricted classes of membranes it has even been shown that the nodal count alone determines the membrane up to isometries, cf. [SmSa,KS,Kl2].

The purpose of this article is to show by a counterexample that this conjecture is false, but we also add some evidence that it may hold after suitable modifications. Our work is concerned with the case of flat tori, and our examples are taken from some isospectral families constructed by Conway and Sloane [CS] in dimension four. These families depend on four real parameters and were conjectured by the authors to be non-isometric whenever no two of the parameters coincide. This has been proved in a recent preprint using lattice theory, cf. [CeHe]. To attack the conjecture analytically, we compute exactly the nodal count for a large class of eigenfunctions on an arbitrary flat torus, the main result being given in Theorem 1.21 below, and we apply this to construct the counterexample in Theorem 3.20. In the example, however, the two isospectral tori are not isometric for trivial reasons, hence the challenge remains to prove the conjecture of Conway and Sloane in the remaining cases by using the nodal count, and we show that this can be done in a number of interesting cases. So far, our method cannot treat the general case but we expect that it can be suitably extended to achieve this goal.

The plan of the paper is as follows. In Sect. 1, we deal with the spectral theory of flat tori in general and prove an explicit formula for the nodal count of what we call "basic eigenfunctions", cf. Theorem 1.21. In Sect. 2, we examine in some detail the four-parameter family of isospectral tori in dimension four of Conway and Sloane [CS]. As the main result of our analysis, we show in Subsect. 2.3 that this family provides infinitely many mutually non-isometric examples of isospectral pairs with the same nodal count. In conclusion, we add some examples of isospectral pairs of flat tori from the same family which are distinguished by their nodal count.

1. Flat Tori and Their Nodal Count

1.1. Generalities. A flat torus is defined by a lattice, Γ, in $\mathbb{R}^m$, $m \in \mathbb{N}$, as the closed manifold

$$T_\Gamma = T := \mathbb{R}^m / \Gamma. \tag{1.1}$$

The projection

$$\pi_\Gamma = \pi : \mathbb{R}^m \to T \tag{1.2}$$

is a covering, hence induces a flat metric, g_{flat}, on T and exhibits Γ as the fundamental group of T; we will consider the flat manifold (T, g_{flat}) in what follows. The dual lattice, Γ^*, of Γ is defined by

$$\Gamma^* := \{\gamma^* \in \mathbb{R}^m : \langle \gamma^*, \gamma \rangle \in \mathbb{Z}, \gamma \in \Gamma\}. \tag{1.3}$$

We may choose a *basis*, $(\gamma_j)_{j=1}^m$, of Γ that is a set of generators of Γ which is a basis of $\mathbb{R}^m$. Then the corresponding dual basis, $(\gamma_i^*)_{i=1}^m$, where $\langle \gamma_i^*, \gamma_j \rangle = \delta_{ij}$, is a basis of Γ^*. In terms of a basis we define the corresponding *fundamental parallelotope* of Γ,

$$\mathcal{F} := \left\{ \sum_{j=1}^m x_j \gamma_j : x_j \in [0, 1) \right\}. \tag{1.4}$$

The Laplacian Δ_T is given locally by the Euclidean Laplacian in $\mathbb{R}^m$; it is an essentially self-adjoint operator in $L^2(T)$ with domain $C^\infty(T)$ and discrete, since T is closed. If ψ denotes any eigenfunction of Δ_T with eigenvalue λ then ψ lifts to $\mathbb{R}^m$ as a solution of the Helmholtz equation,

$$(\Delta_{\mathbb{R}^m} + \lambda)\tilde{\psi} = 0, \quad \tilde{\psi} := \psi \circ \pi.$$

Then there are vectors $\gamma_j^* \in \Gamma^*$ with

$$4\pi^2 |\gamma_j^*|^2 = \lambda,$$

and numbers $\alpha_j, \beta_j \in \mathbb{C}$ with

$$|\alpha_j|^2 + |\beta_j|^2 > 0, \quad j = 1, \ldots, l, \tag{1.5}$$

such that

$$\tilde{\psi}(x) = \sum_{j=1}^l \big(\alpha_j \exp(2\pi i \langle x, \gamma_j^* \rangle) + \beta_j \exp(-2\pi i \langle x, \gamma_j^* \rangle)\big). \tag{1.6}$$

If ψ has the representation (1.6) then we will call it *an eigenfunction of order l.*

The many symmetries of T then allow to describe quite explicitly the spectrum of Δ_T and the isometry class of T. Alternatively, we may translate the problem to the lattice Γ^*, in defining, for $\lambda \in \mathbb{R}_+$, the *representation spaces* of Γ^* by

$$\Gamma^*(\lambda) := \{\gamma^* \in \Gamma^* : |\gamma^*|^2 = \lambda\}, \tag{1.7}$$

and the *representation numbers* of Γ^* by

$$N_{\Gamma^*}(\lambda) := \sharp\, \Gamma^*(\lambda). \tag{1.8}$$

Then the following result is classical, for a proof see e. g. [BGM, Thm.D.8, Prop.B.I.2].

Theorem 1.9. 1. *Two flat tori $T_{\Gamma_1}, T_{\Gamma_2}$ are isospectral if and only if the lattices Γ_1^* and Γ_2^* have the same representation numbers.*
2. *Two flat tori $T_{\Gamma_1}, T_{\Gamma_2}$ are isometric if and only if the lattices Γ_1^* and Γ_2^* are congruent in $\mathbb{R}^m$.*

3. *An orthonormal basis of* $L^2(T_\Gamma)$ *consisting of eigenfunctions of* Δ_{T_Γ} *is given by* $(\psi_{\gamma^*})_{\gamma^* \in \Gamma^*}$, *where*

$$\tilde{\psi}_{\gamma^*}(x) = (\operatorname{vol} \mathcal{F})^{-1/2} \exp\left(2\pi i \langle x, \gamma_j^* \rangle\right). \tag{1.10}$$

As a simple consequence, we see that

$$\operatorname{spec} \Delta_T = \{4\pi^2 |\gamma^*|^2 : \gamma^* \in \Gamma^*\}, \tag{1.11}$$

with eigenspaces

$$E_{\Delta_T}(\lambda) = \operatorname{span}_{\mathbb{C}}\{\psi_{\gamma^*} : \gamma^* \in \Gamma^*(\lambda/4\pi^2)\}. \tag{1.12}$$

Thus, Theorem 1.9 tells us that to solve the inverse spectral problem for flat tori, we have to determine the isometry class of Γ^* from spec T or, alternatively, from the representation numbers of Γ^*. In view of Theorem 1.9, our problem can hence be restated as follows: *is the congruence class of a lattice determined by its representation numbers?*

In this form the question is much older than in its isospectral incarnation (in fact, Milnor's counterexample [M] quoted above was based on a counterexample due to Witt [W], observing the equivalence of the isospectral problem for lattices and flat tori). It is known that the answer is positive in dimensions two [BGM, Ch.III, Prop.B.I.4] and three [Sch] but generally negative in dimensions greater than three (see Theorem 2.65 and Theorem 3.18 below). Thus we may ask whether the nodal count provides sufficient additional information to resolve isospectrality into isometry, at least in the case of flat tori.

1.2. The nodal count. We now fix a flat torus $T = \mathbb{R}^m/\Gamma$, and a general *real eigenfunction*, ψ, with lift $\tilde{\psi}$ given by (1.6); this implies that

$$\beta_j = \overline{\alpha_j}, \tag{1.13}$$

$$\tilde{\psi}(x) = \sum_{j=1}^{l} 2|\alpha_j| \cos\left(2\pi \langle x, \gamma_j^* + \delta_j \rangle\right), \tag{1.14}$$

where $\alpha_j/|\alpha_j| = \exp(2\pi i \delta_j)$. If the vectors $(\gamma_j^*)_{j=1}^l \subset \Gamma^*$ are *linearly independent*, implying $l \le m$, then we can extend $(\gamma_j^*)_{j=1}^l$ to a basis of $\mathbb{R}^m$ somehow and change coordinates by

$$y_j(x) := \langle x, \gamma_j^* \rangle + \delta_j, \quad j = 1, \dots, m, \tag{1.15}$$

such that

$$\tilde{\psi}(y) := \tilde{\psi}(x(y)) = \sum_{j=1}^{l} a_j \cos(2\pi y_j), \tag{1.16}$$

where

$$a_j = |\alpha_j| + |\beta_j| > 0 \quad \text{for all } j.$$

Such eigenfunctions will be called *basic* in what follows, and *simple* if $l = 1$; we will now show that their nodal count can be calculated explicitly. Before stating the corresponding result, Theorem 1.21 below, we prepare a simple lemma.

Lemma 1.17. *Let $\Sigma \subset \mathbb{Z}^l$ be a sublattice of rank l, such that*

$$G := \mathbb{Z}^l / \Sigma$$

is a finite abelian group. If $(\sigma_j)_{j=1}^n$ denotes a set of generators for Σ, then

$$d := \sharp\, G = \gcd\left\{|\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_l}| : I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}\right\} =: \tilde{d}. \quad (1.18)$$

Here, gcd *denotes the greatest common divisor and the norm is induced by the standard metric of $\mathbb{R}^l$.*

Proof. The map $\mathbb{R}^l/\Sigma \to \mathbb{R}^l/\mathbb{Z}^l$, $x + \Sigma \mapsto x + \mathbb{Z}^l$, is a covering map the degree of which is given by the Euclidean volume of any fundamental parallelotope of Σ. Hence, for any basis $(\zeta_j)_{j=1}^l$ of Σ we have

$$d = |\zeta_1 \wedge \ldots \wedge \zeta_l| = \sharp\, G.$$

Using the representations

$$\zeta_j = \sum_{k=1}^n \alpha_{jk}\sigma_k, \quad \sigma_k = \sum_{j=1}^l \beta_{kj}\zeta_j, \quad \alpha_{jk}, \beta_{kj} \in \mathbb{Z},$$

we obtain

$$d = |\sum_{k_1,\ldots,k_l=1}^n \alpha_{1k_1} \ldots \alpha_{lk_l}\, \sigma_{k_1} \wedge \ldots \wedge \sigma_{k_l}|,$$

which shows that $\tilde{d}$ divides d. On the other hand,

$$|\sigma_{k_1} \wedge \ldots \wedge \sigma_{k_l}| = |\sum_{j_1,\ldots,j_l=1}^l \beta_{k_1 j_1} \ldots \beta_{k_l j_l}\; \zeta_{j_1} \wedge \ldots \wedge \zeta_{j_l}| \quad (1.19)$$

$$= |\det(\beta_{k_r j_s})|\, |\zeta_1 \wedge \ldots \wedge \zeta_l|, \quad (1.20)$$

such that d divides $\tilde{d}$, too. □

We can now present the main result of this section.

Theorem 1.21. *Let $(\sigma_j)_{j=1}^n$ be a set of generators of Γ and consider a basic eigenfunction, ψ, in the representation (1.16) determined by l linearly independent vectors $(\gamma_j^*)_{j=1}^l$ in Γ^*.*

1. *If for all $j = 1, \ldots, l$,*

$$a_j < \sum_{k \neq j} a_k, \quad (1.22)$$

then

$$\mathrm{nc}(\psi) = 2. \quad (1.23)$$

2. *If for some* j,

$$a_j > \sum_{k \neq j} a_k, \tag{1.24}$$

or if

$$a_j = \sum_{k \neq j} a_k, \tag{1.25}$$

and $l \geq 3$, *then we have*

$$\mathrm{nc}(\psi) = 2 \gcd \left(\gamma_j^*(\sigma_k)\right)_{k=1,\ldots,n}. \tag{1.26}$$

3. *If* (1.25) *holds and* $l = 2$, *then we obtain*

$$\mathrm{nc}(\psi) = 2 \gcd \left(\det \begin{pmatrix} \gamma_1^*(\sigma_{j_1}) & \gamma_1^*(\sigma_{j_2}) \\ \gamma_2^*(\sigma_{j_1}) & \gamma_2^*(\sigma_{j_2}) \end{pmatrix}\right)_{j_1, j_2=1,\ldots,n}. \tag{1.27}$$

Proof. We remark first that basic eigenfunctions have the same number of positive and of negative nodal domains. Indeed, the representation (1.16) shows that

$$\psi \circ \tau = -\psi,$$

if τ denotes the isometry of T induced by the translation $y_j \mapsto y_j + 1/2,\ j = 1, \ldots, l$. Hence it is enough to count positive nodal domains of basic eigenfunctions.

Next we observe that the group Γ acts on the positive nodal domains of $\tilde{\psi}$ in such a way that the nodal count of ψ equals the cardinality of the orbit space; this is the basic principle of our proof.

1. If $y_{(l)} := (y_1, \ldots, y_l) \in \mathbb{Z}^l$ then, clearly, $\tilde{\psi}(y) > 0$. By deforming each coordinate linearly, we see from (1.22) that the set $\{y_{(l)} \in \mathbb{Z}^l\}$ is contained in a single nodal domain of $\tilde{\psi}$. Moreover, if $\tilde{\psi}(y) > 0$, then we can connect y in $\tilde{\psi}^{-1}(0, \infty)$ to a point $y_{(l)} \in \mathbb{Z}^l$, by deforming each coordinate y_j in the direction of increasing values of $\cos(2\pi y_j)$. Thus $\tilde{\psi}$ has a single positive nodal domain.
2. We may and will assume that $j = 1$. If $r \in \mathbb{Z}$, then on the hypersurface $H_r := \{y_1 = r\}$ we have $\tilde{\psi} \geq 0$. Since $l \geq 3$ in the case (1.25), we see that H_r intersects a single positive nodal domain of $\tilde{\psi}$. Thus we obtain a map from $\mathbb{Z}$ to the positive nodal domains of $\tilde{\psi}$, which is seen to be surjective by deforming as in Part 1, and injective since, by (1.24), $\tilde{\psi}(y) \leq 0$ if $\cos(2\pi y_1) = -1$.
   Now, by (1.15), the action of Γ on $\mathbb{Z}$ induced by the bijection just constructed is given by

   $$\gamma(r) = r + \langle \gamma_1^*, \gamma \rangle, \quad \gamma \in \Gamma,\ r \in \mathbb{Z}.$$

   Thus, if Σ denotes the subgroup of $\mathbb{Z}$ generated by $\left(\gamma_1^*(\gamma)\right)_{\gamma \in \Gamma}$, then the positive nodal count of ψ is given by $\sharp\, \mathbb{Z}/\Sigma$, and the proof of Part 2 is completed by Lemma 1.17.

3. Assume, finally, that (1.25) holds and $l = 2$ such that

$$\tilde{\psi}(y) = a\big(\cos(2\pi y_1) + \cos(2\pi y_2)\big), \ \ a > 0.$$

To label the positive nodal domains of $\tilde{\psi}$ in this case, we note that $\tilde{\psi}(y) = 2a$ if $y \in H_{r,s} := \{y \in \mathbb{R}^m : y_1 = r, \ y_2 = s\}$, $(r, s) \in \mathbb{Z}^2$, which yields, by the arguments used before, a bijection between $\mathbb{Z}^2$ and the positive nodal domains of $\tilde{\psi}$. The ensuing action of Γ on $\mathbb{Z}^2$ is given by

$$(r, s) \mapsto \big(r + \gamma_1^*(\gamma), s + \gamma_2^*(\gamma)\big),$$

and another application of Lemma 1.21 completes the proof. □

We remark in conclusion that Part 1 of the theorem easily shows that there are infinitely many eigenfunctions ψ with $nc(\psi) = 2$, a fact that does not seem to have been stated in the literature in this generality, though a special case has been treated in the dissertation of A. Stern, cf. [CH, p. 396].

Of course, the basic eigenfunctions are very special, and the nodal pattern of a general eigenfunction may be considerably more complicated. It remains a challenge to identify larger classes of eigenfunctions which admit an explicit formula of their nodal count.

2. The Conway-Sloane Four-Parameter Family of Isospectral Tori

2.1. Geometry and spectral theory. We will now deal with an example, the four-parameter family of pairs of isospectral flat tori in dimension four introduced by Conway and Sloane in [CS]; we elaborate here on the material presented in that paper and add a few results. We begin with the following general construction.

Denote by $(e_j)_{j=1}^4$ the standard basis of $\mathbb{R}^4$, orthonormal with respect to the standard scalar product $\langle \cdot, \cdot \rangle$ with norm $|| \cdot ||$. Then we introduce the skew-symmetric 4×4 -matrix,

$$B := (b_{jk}) := \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} =: (B_1, B_2, B_3, B_4), \tag{2.1}$$

with $(B_j)_{j=1}^4$ the column vectors of B. Using B we introduce further matrices by

$$G^{\pm} := \pm 3I + B =: (g_{jk}^{\pm}), \tag{2.2}$$
$$S^{\pm} := \mp 3I + 3B =: (s_{jk}^{\pm}). \tag{2.3}$$

Note that $(B_j)_{j=1}^4$ is an orthogonal basis of $\mathbb{R}^4$ with $||B_j||^2 = 3$ for all j; consequently,

$$B^2 = -3I, \tag{2.4}$$
$$G^+G^- = -12I, \tag{2.5}$$
$$S^+S^- = -36I. \tag{2.6}$$

It is convenient to bring in the orthogonal matrix

$$O := \frac{1}{2}(I + B) \tag{2.7}$$

such that e.g.

$$S^{\pm} = \mp 6\, O^{\mp 1}, \tag{2.8}$$
$$G^{\pm} B = -6\, O^{\mp 1}. \tag{2.9}$$

Now let $(f_j)_{j=1}^4$ be any basis of $\mathbb{R}^4$. Then we introduce four new bases by

$$\gamma_j^{\pm} := \sum_k g_{jk}^{\pm} f_k, \tag{2.10}$$
$$\sigma_j^{\pm} := \sum_k s_{jk}^{\pm} f_k; \tag{2.11}$$

for a given vector $x \in \mathbb{R}^4$ we will distinguish the respective coefficients by writing

$$x = \sum_j n_j f_j, \tag{2.12}$$
$$= \sum_j m_j^{\pm} \gamma_j^{\pm}, \tag{2.13}$$
$$= \sum_j l_j^{\pm} \sigma_j^{\pm}. \tag{2.14}$$

Now we can introduce the main objects of study in the remaining part of this paper.

Definition 2.15. *We denote by* $\Gamma^{\pm}$ *and* $\Sigma^{\pm}$ *the lattices generated in* $\mathbb{R}^4$ *by* $(\gamma_j^{\pm})_{j=1}^4$ *and* $(\sigma_j^{\pm})_{j=1}^4$*, respectively.*

Lemma 2.16. *We have the following relations:*

$$\sigma_j^{\pm} = \sum_{k=1}^{4} (\pm b_{jk}) \gamma_k^{\pm}, \tag{2.17}$$
$$m^{\pm} = \mp B l^{\pm}, \tag{2.18}$$
$$n = -G^{\mp} m^{\pm} = \mp 6\, O^{\pm 1} l^{\pm}, \tag{2.19}$$
$$m^{-} = -O^{\dagger} m^{+}, \tag{2.20}$$
$$l^{-} = O^{\dagger} l^{+}. \tag{2.21}$$

Proof. To prove (2.17) we compute with (2.4)

$$\pm B G^{\pm} = \pm B(\pm 3I + B) = (\mp 3I + 3B) = S^{\pm}.$$

The remaining relations are proved similarly. □

It is convenient to also use the notation

$$\tilde{O} := 2O = I + B. \tag{2.22}$$

It follows from this lemma that $\Sigma^{\pm} \subset \Gamma^{\pm}$ such that we can decompose $\Gamma^{\pm}$ as union of its cosets modulo $\Sigma^{\pm}$. For integer vectors $m, n, p \in \mathbb{Z}^k$ we will employ the notation

$$m = n\,(p) \Leftrightarrow m_j \equiv n_j \bmod p, \quad j = 1, \ldots, k. \tag{2.23}$$

Corollary 2.24. *If $\gamma \in \Gamma^{\pm}$ then*

$$\gamma \in \Sigma^{\pm} \Leftrightarrow Bm^{\pm}(\gamma) = 0 \ (3). \tag{2.25}$$

Consequently, we have the coset decomposition

$$\Gamma^{\pm} = \bigsqcup_{j=-4}^{j=4} \Gamma_j^{\pm}, \tag{2.26}$$

where

$$\Gamma_0^{\pm} := \Sigma^{\pm}, \tag{2.27}$$

$$\Gamma_j^{\pm} := (\operatorname{sgn} j)\gamma_{|j|}^{\pm} + \Sigma^{\pm}, \quad j \neq 0. \tag{2.28}$$

Proof. It follows from (2.18) that $\gamma \in \Gamma^{\pm}$ is actually in $\Sigma^{\pm}$ if and only if

$$3l^{\pm}(\gamma) = \pm Bm^{\pm}(\gamma) \in 3\mathbb{Z} \ \Leftrightarrow \ \pm Bm^{\pm}(\gamma) = 0 \ (3), \tag{2.29}$$

such that $\left((\operatorname{sgn} j)\,\gamma_j^{\pm}\right)_{|j|\leq 4}$ is a set of generators for the group $\Gamma^{\pm}/\Sigma^{\pm}$. Then an easy inspection shows that

$$\gamma_j^{\pm} = \varepsilon\gamma_k^{\pm} \bmod \Sigma^{\pm}, \quad 1 \leq j, k \leq 4, \ |\varepsilon| = 1,$$

is equivalent to $j = k$, $\varepsilon = 1$, which completes the proof. □

The lattices $\Gamma^{\pm}$ and $\Sigma^{\pm}$ contain the common sublattices $\Gamma^{\neq} := \Gamma^{+} \cap \Gamma^{-}$ and $\Sigma^{\neq} := \Sigma^{+} \cap \Sigma^{-}$, respectively, which are easily characterized as follows, using the notation

$$J(k) := \sum_{i=1}^{4} k_i. \tag{2.30}$$

Lemma 2.31. 1. *A vector $\gamma = \sum_j m_j^{\pm}\gamma_j^{\pm} \in \Gamma^{\pm}$ is in $\Gamma^{\mp}$ if and only if*

$$J(m^{\pm}) \text{ is even.} \tag{2.32}$$

2. *A vector $\sigma = \sum_j l_j^{\pm}\sigma_j^{\pm} \in \Sigma^{\pm}$ is in $\Sigma^{\mp}$ if and only if*

$$J(l^{\pm}) \text{ is even.} \tag{2.33}$$

3.

$$\Sigma^{\pm} \cap \Gamma^{\neq} = \Sigma^{\neq}. \tag{2.34}$$

Proof. 1. We have the coefficient transformation (cf. (2.20))

$$m^{-} = -O^{\dagger}m^{+}, \quad m^{+} = -Om^{-}.$$

By definition, $o_{ij} =: \frac{1}{2}\tilde{o}_{ij}$ with $|\tilde{o}_{ij}| = 1$ for all i, j such that

$$J_i(m^{+}) := \sum_j \tilde{o}_{ij} m_j^{+} \tag{2.35}$$

has the same parity as $J(m^{+})$ for all i. Hence $m^{-} \in \mathbb{Z}^4$ if and only if $J(m^{+})$ is even, and vice versa.

2. Noting that
$$l^+ = Ol^-, \quad l^- = O^\dagger l^+,$$
the proof is the same as in Part 1.
3. We write
$$\gamma = \sum_j m_j^\pm \gamma_j^\pm = \sum_j l_j^\pm \sigma_j^\pm,$$
and we have to show that $J(m^\pm)$ even implies $J(l^\pm)$ even. But (2.18) gives $l^\pm = \pm\frac{1}{3}Bm^\pm$, hence
$$3J(l^\pm) = \pm \sum_{j,k} b_{jk} m_k^\pm =: \sum_k \underline{b}_k m_k^\pm.$$
Since all $\underline{b}_k$ are odd, the assertion follows. □

In what follows, we also need a particular group of isometries of $\mathbb{R}^4$. Denote by $R \subset O(4)$ the finite abelian group generated by the reflections $\{\tau_j\}_{j=1}^4$ in the hyperplanes $n_j = 0$,
$$\tau_j f_k := (1 - 2\delta_{jk}) f_k; \tag{2.36}$$
we note that
$$\tau_j \gamma_j^\pm = \gamma_j^\mp. \tag{2.37}$$
We denote by R^+ the subgroup of orientation preserving elements of R. Then we have the following facts.

Lemma 2.38. 1. *For all j we have*
$$\tau_j(\Sigma^\pm) \subset \Sigma^\mp; \tag{2.39}$$
in particular, the lattices Σ^+ and Σ^- are congruent.
2. *If for some $\rho \in R$ and some i we have*
$$\rho(\gamma_i^\pm) \in \Gamma^+ \cup \Gamma^-, \tag{2.40}$$
then
$$\rho = \pm I \quad \textit{or} \quad \rho = \pm\tau_i. \tag{2.41}$$

Proof. 1. Denote by T_j the matrix of τ_j in the basis (f_j). Observe now that the rows of $\tilde{O}$ and their negatives give all vectors $\varepsilon = (\varepsilon_j)_{j=1}^4 \in \{-1, 1\}^4$ with the property
$$\xi(\varepsilon) := \prod_j \varepsilon_j = -1,$$
such that the rows of $\tilde{O}^\dagger$ and their negatives yield all vectors $\varepsilon \in \{-1, 1\}^4$ with the property $\xi(\varepsilon) = 1$. It follows that
$$T_j \tilde{O}^{\pm 1} = \tilde{O}^{\mp 1} P_j, \tag{2.42}$$
for a certain *permutation matrix* $P_j = (\delta_{r,\sigma_j(s)}\alpha_j(s))_{rs} =: P_{\sigma_j,\alpha_j}$, where $\sigma_j \in S_4$, viewed as the group of bijections of $\mathbb{N}_4 = \{1, 2, 3, 4\}$, and $\alpha : \mathbb{N}_4 \to \{-1, 1\}$. Equation (2.42) implies the assertion in view of (2.19).

2. Consider an element $\rho \in R$, such that $\rho f_k =: \varepsilon_k f_k$ for certain numbers $\varepsilon_k \in \{-1, 1\}$. The property (2.40) together with Corollary 2.24 implies the relation

$$\rho \gamma_i^{\xi} = \alpha \gamma_j^{\eta} + \sigma,$$

with $\xi, \eta \in \{+, -\}$, $j = j(\rho, i, \xi)$, $\alpha \in \{-1, 0, 1\}$, and $\sigma \in \Sigma^{\eta}$. Calculating mod 3 in the n- coordinates, we find with (2.19) and Corollary 2.24 the identities

$$\varepsilon_k b_{ki} = \alpha b_{kj} \ (3),$$

for all k. If $i \neq j$, we put $k = i$ and obtain $\alpha = 0$, contradicting (2.25). Hence we must have $i = j$ which gives $|\alpha| = 1$ and $\varepsilon_k = \alpha$ for $k \neq i$ and completes the proof. □

We remark for later use that each σ_j is a 3-cycle and hence even, and that the permutation matrices form a group under multiplication, satisfying the rules

$$P_{\sigma_1,\alpha_1} P_{\sigma_2,\alpha_2} = P_{\sigma_1\sigma_2,(\sigma_2^*\alpha_1)\alpha_2}, \tag{2.43}$$

$$P_{\sigma,\alpha}^{-1} = P_{\sigma^{-1},(\sigma^{-1})^*\alpha}; \tag{2.44}$$

$$(P_{\sigma,\alpha})^{\dagger} = P_{\sigma^{-1},((\sigma^{-1})^*\alpha)^{-1})}. \tag{2.45}$$

We turn to the spectral theory of the Laplacians, $\Delta^{\pm}$, defined on the flat tori $T^{\pm}$ which are *associated to the dual lattices of* $\Gamma^{\pm}$; we will express the results in terms of the lattices $\Gamma^{\pm}$. For this, we have to specify the metric properties of the basis (f_j). We follow [CS] in *assuming from now on that our basis is orthogonal*, i.e. that we have

$$\langle f_j, f_k \rangle = \delta_{jk} a_j^2, \tag{2.46}$$

for certain positive numbers $(a_j)_{j=1}^4$. Thus if $\gamma = \sum_{j=1}^4 n_j f_j \in \Gamma^{\pm}$ then

$$\psi_\gamma \in E^{\pm}(\lambda) := \ker(\Delta^{\pm} - \lambda I) \Leftrightarrow \tag{2.47}$$

$$|\gamma|^2 = \sum_j n_j^2 a_j^2 = (4\pi^2)^{-1}\lambda \Leftrightarrow \tag{2.48}$$

$$\gamma \in \Gamma^{\pm}(\lambda). \tag{2.49}$$

Since γ and $-\gamma$ have the same length, the cardinality of $\Gamma^{\pm}(\lambda)$ is even and at least 2 for positive eigenvalues.

Using the group R introduced above we define the set, for $\gamma \in \Gamma^{\pm}(\lambda)$,

$$\Gamma_\gamma^{\pm}(\lambda) := R\gamma \cap \Gamma^{\pm}, \tag{2.50}$$

such that

$$\Gamma_\gamma^{\pm}(\lambda) \subset \Gamma^{\pm}(\lambda).$$

The following result is obvious.

Lemma 2.51. *If the numbers $(a_j^2)_{j=1}^4$ are linearly independent over $\mathbb{Q}$ then for $\gamma \in \Gamma^{\pm}(\lambda)$,*

$$\Gamma^{\pm}(\lambda) = \Gamma_{\gamma}^{\pm}(\lambda), \tag{2.52}$$

and

$$\sharp \left\{ \Gamma_{\gamma}^{+}(\lambda) \cup \Gamma_{\gamma}^{-}(\lambda) \right\} \leq 16. \tag{2.53}$$

The spaces $\Gamma_{\gamma}^{\pm}(\lambda)$ can be described quite explicitly.

Theorem 2.54. 1. *If $\gamma \in \Gamma_j^{\pm}$, $j \neq 0$, then*

$$\Gamma_{\gamma}^{\pm}(\lambda) = \{\gamma, -\gamma\}, \tag{2.55}$$
$$\Gamma_{\gamma}^{\mp}(\lambda) = \{\tau_j \gamma, -\tau_j \gamma\}. \tag{2.56}$$

2. *If $\gamma \in \Sigma^{\neq}$, then*

$$\Gamma_{\gamma}^{+}(\lambda) = \Gamma_{\gamma}^{-}(\lambda). \tag{2.57}$$

3. *If $\gamma \in \Sigma^{\pm} \backslash \Sigma^{\neq}$, then*

$$\Gamma_{\gamma}^{\pm}(\lambda) = R^{+}\gamma, \quad \Gamma_{\gamma}^{\mp}(\lambda) = \tau_1 R^{+}\gamma, \tag{2.58}$$

and

$$\sharp\ \Gamma_{\gamma}^{\pm}(\lambda) = \sharp\ \Gamma_{\gamma}^{\mp}(\lambda) = 8. \tag{2.59}$$

Proof. 1. The statement follows immediately from Lemma 2.38, Part 2.
2. This statement follows from Lemma 2.38, Part 1.
3. For $\gamma \in \Sigma^{+} \backslash \Sigma^{\neq}$ we have from (2.21) and Lemma 2.31, Part 2,

$$n_i(\gamma) = (-6\, Ol^{+}(\gamma))_i = -3 \sum_j \tilde{o}_{ij} l^{+}(\gamma)_j = J(l^{+}(\gamma)) = 1\ (2),$$

since $|\tilde{o}_{ij}| = 1$ for all i, j; in particular, $n_i(\gamma) = 0$ for all i. The proof for $\gamma \in \Sigma^{-}$ is similar. This together with (2.39) proves the assertion. □

We now discuss the relationship between the geometry and the spectral theory associated with the lattices Γ^{+} and Γ^{-}. For the geometry, we get the following result.

Lemma 2.60. 1. *If $a_i = a_j$ for $i \neq j$ then Γ^{+} and Γ^{-} are congruent and the associated tori T^{+} and T^{-} are isometric.*
2. *If the numbers $(a_i^2)_{i=1}^4$ are linearly independent over $\mathbb{Q}$ then the lattices Γ^{+} and Γ^{-} are not congruent.*

Proof. 1. We have to show that there are matrices $M \in GL(4, \mathbb{Z})$ and $O \in O(4)$ such that $\det M = \pm 1$ and

$$G^{-}\Delta(a) = MG^{+}\Delta(a)O, \tag{2.61}$$

where $\Delta(a) := (\delta_{lk} a_k)$. Now it is easily verified that we can find, for all i, j, a permutation matrix $P_{ij} = P_{(ij),\alpha_{ij}}$, with (ij) the transposition commuting i and j and $\alpha_{ij}(i) = \alpha_{ij}(j) = 1$, such that

$$BP_{ij} = -P_{ij}B;$$

cf. the remark after Lemma 2.39. Then (2.61) is satisfied with $P_{ij} =: M =: O$.

2. Arguing by contradiction, we assume that Γ^+ and Γ^- are congruent. Then there is $O \in O(4)$ with

$$O(\Gamma^{\pm}) = \Gamma^{\mp}. \tag{2.62}$$

If $O = (o_{ij})$ with respect to $(f_k)_k$ then we find

$$a_i^2 = \sum_j o_{ij}^2 a_j^2, \tag{2.63}$$

for all i. Now, from (2.62) and (2.5) we have with some $T \in PGL(4, \mathbb{Z})$,

$$-12 o_{ij} = (G^- T G^-)_{ij} \in \mathbb{Q},$$

and by linear independence over $\mathbb{Q}$ we derive from (2.63)

$$o_{ij}^2 = \delta_{ij}.$$

Hence $O \in R$ and the proof is completed by applying Lemma 2.38, Part 2. $\square$

On the basis of these facts, Conway and Sloane conjectured that

$$\text{the lattices } \Gamma^+ \text{ and } \Gamma^- \text{ are not congruent if } a_i \neq a_j,\ i \neq j, \tag{2.64}$$

and they verified this conjecture for the classically integral lattices among them with determinant $(a_1 a_2 a_3 a_4)^2 \leq 1000$. Recently, Cervino and Hein proved the conjecture in full generality using their theory of lattice invariants [CeHe].

Nevertheless, the lattices Γ^+ and Γ^- are always isospectral which is the key observation of Conway and Sloane. This fact has become much more transparent due to their geometric explanation of these lattice pairs; in fact, as an easy consequence of Lemma 2.38 we find

Theorem 2.65. *For all choices of the numbers $(a_i)_1^4$, the lattices Γ^+ and Γ^- are isospectral. In fact, the map*

$$\Psi : \Gamma^+ \to \Gamma^-,$$

defined by

$$\Psi | \Gamma_0 := \tau_1, \tag{2.66}$$

$$\Psi | \Gamma_j := \tau_j, \quad j \neq 0, \tag{2.67}$$

maps $\Gamma^+(\lambda)$ bijectively to $\Gamma^-(\lambda)$, for all $\lambda \in \mathbb{R}_+$.

3. Isonodality

We now turn to the question whether isonodality resolves isospectrality in the example just studied, i. e. whether the tori T^+ and T^- can be distinguished by their nodal count.

For tori, the notion of isonodality as defined in (0.6), (0.7) can be refined as follows. We call the tori T^+ and T^- *isonodal of order* p, $p \in \mathbb{N}$ if there is a length preserving bijection

$$\Psi_p : \Gamma^+ \to \Gamma^-$$

such that for all $\lambda > 0$, the pair of eigenfunctions

$$\sum_{j=1}^{p} \alpha_j \psi_{\gamma_j} \quad \text{and} \quad \sum_{j=1}^{p} \alpha_j \psi_{\Psi_p(\gamma_j)}$$

has the same nodal count for any choice of $(\gamma_j) \subset \Gamma^+(\lambda)$ and $(\alpha_j) \subset \mathbb{C}$. Then one may hope that already the nodal count of order one may distinguish the tori T^+ and T^-.

In Subsect. 3.1 we show that this is not the case. Subsection 3.2 deals with isonodality in general; we show that even the full nodal count does not distinguish T^+ and T^- if the numbers $(a_j^2)_j$ are linearly independent over $\mathbb{Q}$, thus disproving the conjecture mentioned in the Introduction, in view of Lemma 2.60. Finally, in Subsect. 3.3 we add some evidence that the nodal count does distinguish the tori if we allow general parameters $(a_j^2)_j$.

3.1. Isonodality of order one. We prove the following result.

Theorem 3.1. *The tori T^+ and T^- are isonodal of order one.*

Proof. The theorem will be proved if we construct a bijective map

$$\Psi_1 : \Gamma^+ \to \Gamma^-$$

which preserves the length and the nodal count. We will use for $\gamma \in \Gamma^\pm$ the coordinates $n^\pm, m^\pm, l^\pm$ in the respective bases, as before, and we will also use the abbreviation

$$[m] := \gcd (m_j)_j, \tag{3.2}$$

for $m \in \mathbb{Z}^4$. Then we need to construct Ψ_1 in such a way that, for all γ,

$$|n(\Psi_1(\gamma))| = |n(\gamma)|, \quad [m^-(\Psi_1(\gamma))] = [m(\gamma)].$$

We begin by decomposing

$$\Gamma^+ = \Gamma^{\neq} \cup \bigcup_j \dot{\Gamma}_j^+, \tag{3.3}$$

where $\dot{\Gamma}_j^+ := \Gamma_j^+ \backslash \Gamma^{\neq}$.

Case 1. $\gamma \in \Gamma^{\neq}$. Clearly, we obtain an isospectral and isonodal bijection $\Psi_1 : \Gamma^{\neq} \to \Gamma^{\neq}$ by defining

$$\Psi_1 | \Gamma^{\neq} := I_{\Gamma^{\neq}}. \tag{3.4}$$

Case 2: $\gamma \in \dot{\Gamma}_j^+$, $j \neq 0$. Now we define a bijection from $\dot{\Gamma}_j^+$ to $\dot{\Gamma}_j^-$ by

$$\Psi_1 | \dot{\Gamma}_j := \Psi | \dot{\Gamma}_j^+ = \tau_{|j|} | \dot{\Gamma}_j^+. \tag{3.5}$$

The assertion is a consequence of the following result.

Lemma 3.6. *If* $\gamma \in \dot{\Gamma}_j^{\pm}$, $j \neq 0$, *then*

$$[m^{\pm}(\gamma)] = [n(\gamma)]. \tag{3.7}$$

Proof. All coordinates will be taken for γ which we will suppress in the following formulas. From (2.19), we see that $[m^{\pm}]|[n]$.

From (2.19) and (2.5) we get the relation

$$G^{\pm}n = 12m^{\pm},$$

hence it is enough to prove that 2 and 3 are not common divisors of $(n_j)_{j=1}^4$.

As in the proof of Theorem 2.54, Part 3, we deduce from the assumption $\gamma \notin \Gamma^{\neq}$ that all $n_k^{\pm}$ are odd; this rules out the common divisor 2.

Assume next that

$$n = 3\bar{n}, \quad \bar{n} \in \mathbb{Z}^4.$$

Invoking (2.19) once more we obtain

$$-Bm^{\pm} = 3(\bar{n} \mp m^{\pm}),$$

which yields with (2.4)

$$m^{\pm} = B(\bar{n} \mp m^{\pm}) =: \mp Bl^{\pm}.$$

Since $l^{\pm} \in \mathbb{Z}^4$, it follows from (2.18) that $\gamma^{\pm} \in \Sigma^{\pm}$, contradicting the assumption and completing the proof. $\square$

Case 3. $\gamma \in \dot{\Gamma}_0^+ = \Sigma^+ \backslash \Sigma^{\neq}$. This case is more subtle since it may happen that $[m^+(\gamma)] = 1$ but $[m^-(\tau_j\gamma)] = 3$ for some j. Now it follows from (2.42) and (2.43) through (2.45) that R^+ acts on $\dot{\Gamma}_0^{\pm}$ while each τ_j will map $\dot{\Gamma}_0^{\pm}$ to $\dot{\Gamma}_0^{\mp}$. Hence it is reasonable to look simultaneously at the orbit

$$R^+\gamma = \{\pm\tau_1\tau_j\gamma\}_{j=1}^4 \subset \Sigma^+,$$

and its image in Σ^- under any of the τ_j,

$$R^-\gamma = \{\pm\tau_j\gamma\}_{j=1}^4.$$

We parametrize $\dot{\Gamma}_0^+$ by $l^+ \in \mathbb{Z}^4$ satisfying $J(l^+) = 1\,(2)$. We need the following result.

Lemma 3.8. 1. *For* $\gamma \in \dot{\Gamma}_0^{\pm}$ *with* $[l^{\pm}(\gamma)] = 1$ *the following conditions are equivalent.*

$$[m^{\pm}(\gamma)] = 3; \tag{3.9}$$

$$l^{\pm}(\gamma) = \varepsilon B_i\ (3) \text{ for some } i \in \mathbb{N}_4 \text{ and } \varepsilon \in \{-1, 1\}; \tag{3.10}$$

$$\gamma \in 3\dot{\Gamma}_j^{\pm} \text{ for some } j \neq 0. \tag{3.11}$$

2. *For* $\gamma \in \dot{\Gamma}_0^{\pm}$ *with* $[l^{\pm}(\gamma)] = 1$ *we have*

$$\sharp\left(R^+\gamma \cap \bigcup_{j\neq 0} 3\dot{\Gamma}_j^{\pm}\right) = \sharp\left(R^-\gamma \cap \bigcup_{j\neq 0} 3\dot{\Gamma}_j^{\mp}\right) \in \{0, 2\}. \tag{3.12}$$

Proof. 1. Recall that B_j denotes the j^{th} column of the matrix B. If (3.9) holds then

$$m^{\pm} = \mp B l^{\pm} = 0\,(3),$$

and we may assume that $|l_j^{\pm}| \leq 1$ for all j. Thus we get

$$m^{\pm} = 3\bar{m}^{\pm}, \text{ for some } \bar{m}^{\pm} \in \mathbb{Z}^4, \tag{3.13}$$

hence

$$l^{\pm} = \pm B\bar{m}^{\pm}.$$

This implies

$$4 \geq |l^{\pm}|^2 = 3\sum_j (\bar{m}_j^{\pm})^2 \geq 1, \tag{3.14}$$

hence (3.10).
If (3.10) is satisfied then we have for some $\tilde{l}^{\pm} \in \mathbb{Z}^4$,

$$m^{\pm} = \mp B(\varepsilon B_j + 3\tilde{l}^{\pm}) = \pm 3\varepsilon(e_j + B\tilde{l}^{\pm}),$$

hence, for some $\sigma^{\pm} \in \Sigma^{\pm}$,

$$\gamma^{\pm} =: \pm 3\varepsilon\left(\gamma_j^{\pm} + \sigma^{\pm}\right),$$

which gives (3.11). Finally, (3.11) clearly implies (3.9) which completes the proof.
2. Note first that the equivalent conditions in Part 1 are satisfied by γ and $-\gamma$ simultaneously, and that

$$\sharp\, R^{\pm}\gamma = 8,$$

cf. the proof of Lemma 3.6, Part 1. Assume next that $\gamma \in \dot{\Gamma}_0^{\pm} \cap 3\dot{\Gamma}_i^{\pm}$ and that for some $j \neq k$ and some l we have $\tau_j\tau_k\gamma \in 3\dot{\Gamma}_l^{\pm}$. Denoting by T_s the matrix of τ_s in the n-coordinates as before, we find from (3.10) and (2.19) the relation

$$T_k T_j B_i = \pm B_l\,(3),$$

which is easily seen to be impossible.

Finally, if $\gamma \in \dot{\Gamma}_j^{\pm}$ then, clearly, $\tau_j\gamma \in \dot{\Gamma}_j^{\mp}$, completing the proof of Part 2. □

Now we can define Ψ_1 also for $\gamma \in \dot{\Gamma}_0^{\pm}$. If

$$\sharp\left(R^{+}(\gamma/[l^{\pm}(\gamma)]) \cap \bigcup_{j\neq 0} 3\dot{\Gamma}_j^{\pm}\right) = 0, \tag{3.15}$$

then we put for $\rho \in R^{+}$,

$$\Psi_1\rho\gamma := \tau_1\rho\gamma.$$

If, however,

$$\sharp\left(R^{+}(\gamma/[l^{\pm}(\gamma)]) \cap \bigcup_{j\neq 0} 3\dot{\Gamma}_j^{\pm}\right) = 2, \tag{3.16}$$

then we may assume that

$$\gamma \in [l^{\pm}(\gamma)]3\dot{\Gamma}_j^{\pm}$$

for some $j \neq 0$, and we define for $\rho \in R^+$

$$\Psi_1 \rho\gamma := \tau_j \rho\gamma.$$

This completes the proof of Theorem 3.1. $\square$

3.2. Isonodality of order two: The counterexamples. In this subsection we show that the Conway-Sloane family of isospectral flat 4-tori, treated in Sect. 2, provides counterexamples to the conjecture that the nodal count resolves isospectrality. In fact, this is the case whenever

$$\text{the numbers } (a_j^2)_{j=1}^4 \text{ are linearly independent over } \mathbb{Q}. \tag{3.17}$$

We collect what we have already shown in this case in the following statement which follows from Lemma 2.60, Theorem 2.54, and Theorem 2.65.

Theorem 3.18. *Assume the condition* (3.17)*; then the following assertions hold.*

1. *The flat tori T^+ and T^- are not isometric.*
2. *The eigenspaces of $T^{\pm}$ are given by*

$$E^{\pm}(\lambda) = \{\psi_\gamma : \gamma \in R\gamma \cap \Gamma^{\pm}(\lambda) = \Gamma_\gamma^{\pm}(\lambda)\},$$

and

$$\Gamma_\gamma^+(\lambda) = \begin{cases} \{\pm\gamma\}, & \gamma \in \bigcup_{j\neq 0} \dot{\Gamma}_j^+, \\ \{\pm\tau_j\tau_1\gamma\}_{j=1}^4, & \gamma \in \dot{\Gamma}_0^+, \\ \Gamma_\gamma^-(\lambda), & \gamma \in \Gamma^{\neq}. \end{cases} \tag{3.19}$$

3. *The map Ψ_1 constructed in Theorem 3.1 induces a bijection $E^+(\lambda) \to E^-(\lambda)$ preserving the nodal count of order one, for all $\lambda \in \operatorname{spec} T^+$.*

We now want to prove the following result.

Theorem 3.20. *Under the condition* (3.17)*, the tori T^+ and T^- are isospectral and isonodal.*

To prove this theorem, we have to define a bijection

$$\Psi_2 : E^+(\lambda) \to E^-(\lambda),$$

which preserves the nodal count, by (0.5). In view of Theorem 1.21, this is achieved already by the map Ψ_1 constructed in Theorem 3.1 if $\sharp\,\Gamma^+(\lambda) \leq 2$. Hence we may assume that

$$\Gamma^+(\lambda) = R^+\gamma_1$$

for some $\gamma_1 \in \dot{\Gamma}_0^+$, and we write $\gamma_j := \tau_j \tau_1 \gamma_1$, $j \in \mathbb{N}_4$. Then $\sharp\, \Gamma^+(\lambda) = 8$ and all eigenfunctions in $E^+(\lambda)$ are basic, by Theorem 1.21. Thus we can write the general element of $E^+(\lambda)$ as

$$\psi = \sum_{j=1}^{4} \left(b_{\gamma_j} \psi_{\gamma_j} + b_{-\gamma_j} \psi_{-\gamma_j} \right).$$

It is then natural to introduce

$$\Psi_2 \psi := \sum_{j=1}^{4} \left(b_{\gamma_j} \psi_{\Psi_1 \gamma_j} + b_{-\gamma_j} \psi_{-\Psi_2 \gamma_j} \right), \tag{3.21}$$

since we see, again from Theorem 1.21, that

$$\mathrm{nc}(\psi) = \mathrm{nc}(\Psi_2 \psi),$$

provided that

$$c(\psi) := \sharp\, \{ j \in \mathbb{N}_4 : a_\gamma := a_j := |b_{\gamma_j}|^2 + |b_{-\gamma_j}|^2 > 0 \} \neq 2.$$

In case $c(\psi) = 2$, i.e. we can continue to use the same definition unless

$$a_1 = a_2.$$

Then the nodal count of ψ is given by $[m^+(\gamma_1) \wedge m^+(\gamma_2)]$ which need not coincide with $[m^+(\Psi_1\gamma_1) \wedge m^+(\Psi_1\gamma_2)]$. But in view of our definition of isonodality, cf. (0.5), it suffices to show that the sequences

$$([m^+(\gamma_1) \wedge m^+(\gamma_2)])_{\gamma_1, \gamma_2 \in R^+\gamma_0}$$

and

$$([m^-(\Psi_1\gamma_1) \wedge m^-(\Psi_1\gamma_2)])_{\gamma_1, \gamma_2 \in R^+\gamma_0}$$

coincide up to a permutation.

To achieve this, we parametrize $\dot{\Gamma}_0^+$ by the set $\{ l \in \mathbb{Z}^4 : J(l) = 1\,(2) \}$. In Table 1, we have listed elements $m^{\pm}_{(j)}(l)$ which, together with their negatives, give the m-coordinates of the orbit $R^{\pm}\gamma_0$, in terms of $l = l^+(\gamma) \in \mathbb{Z}^4$. To show that the sequences $\left([m^+_{(j)}(l) \wedge m^+_{(k)}(l)]\right)_{1 \le j < k \le 4}$ and $\left([m^-_{(j)}(l) \wedge m^+_{(k)}(l)]\right)_{1 \le j < k \le 4}$ coincide up to a permutation, we compare them in Tables 2, 3, and 4 in pairs which already show the desired property; we have denoted these subsets by $\left(p^{\pm}_{(2j-1)}(l), p^{\pm}_{(2j)}(l)\right)$, where $j = 1, 2, 3$. The tables then show that only certain specific quadratic forms in the variables l_j occur in the formation of the two strings of six wedge products. The following lemma exploits this and will prove Theorem 3.20, by inspecting the tables.

Lemma 3.22. *Let $a, b, c, d \in \mathbb{Z}$ be given with*

$$ab + cd = 1\ (2), \tag{3.23}$$

$$ac + bd = 0\ (2), \tag{3.24}$$

and define

$$A := ab, \qquad B := cd, \tag{3.25}$$
$$2C := ac - bd, \quad 2D := ac + bd. \tag{3.26}$$

Let

$$\alpha := [A, B, C, D] \tag{3.27}$$

be the greatest common divisor. Then the functions

$$\phi_1 : \{-1, 1\} \ni j \mapsto [A + B + jC, 3\alpha] \in \mathbb{Z}, \tag{3.28}$$
$$\phi_2 : \{-1, 1\} \ni j \mapsto [A - B + jD, 3\alpha] \in \mathbb{Z}, \tag{3.29}$$

are equivalent (in the sense that $\phi_2 = \phi_1 \circ \psi$ for some bijection ψ of $\{-1, 1\}$).

Proof. Let us remark first that the symmetries

$$a \leftrightarrow c, \qquad b \leftrightarrow d, \tag{3.30}$$
$$a \leftrightarrow b, \qquad c \leftrightarrow d, \tag{3.31}$$
$$a \leftrightarrow -a \quad \text{or} \quad b \leftrightarrow -b, \tag{3.32}$$

leave the problem invariant in the sense that they replace ϕ_1 and ϕ_2 by a pair which is equivalent if and only if ϕ_1 and ϕ_2 are equivalent.

Next we observe that α is odd since it divides $A+B = ab+cd$. It follows that any prime divisor, p, of α divides the numbers ab, cd, ac, bd hence w.l.o.g. $a =: pa'$, $d =: pd'$. Thus, if the assertion of the lemma holds for a', b, c, d' then it also holds for a, b, c, d such that it is enough to treat the case

$$\alpha = 1. \tag{3.33}$$

Now it is readily computed that

$$A + B \pm C = a(b \mp c) \pm d(b \pm c) \ (3), \tag{3.34}$$
$$A - B \pm D = a(b \pm c) \mp d(b \pm c) \ (3). \tag{3.35}$$

It remains to discuss two cases.

Case 1. $b^2c^2 = 1 \ (3)$. We then may assume w.l.o.g. that $b = c \ (3)$, and we find

$$A + B + C = -bd \ (3), \quad A + B - C = -ba, \ (3) \tag{3.36}$$
$$A - B + D = bd \ (3), \quad A - B - D = -ba \ (3), \tag{3.37}$$

proving the lemma in this case.

Case 2. $b^2c^2 = 0 \ (3)$. If $b = c = 0 \ (3)$ then clearly $A + B \pm C = A - B \pm D = 0 \ (3)$. Otherwise, w.l.o.g. we may assume $b = 1, c = 0 \ (3)$ which gives

$$A + B \pm C = b(a \pm d), \quad A - B \pm D = b(a \mp d). \tag{3.38}$$

This settles also the second case and completes the proof of the lemma. □

3.3. Some remarks concerning the general case. The general case, where condition (3.17) is not assumed, seems to be much more sensible to nodal counting than the "rigid" situation providing our counterexamples. So far, we have only sporadic evidence corroborating this statement which, nevertheless, has some interest. The highest "degeneration" should occur if $(a_i^2)_{i=1}^4 \subset \mathbb{N}$ in which case we can test isonodality numerically for each concrete choice of the parameters. A typical result is the following.

Theorem 3.39. *Assume that $(a_i^2)_{i=1}^4 \subset \mathbb{N}$, and that $a_1^2 < a_2^2 < a_3^2 < a_4^2 < 20$. Then the associated tori T^+ and T^- are not isonodal of order two, hence are not isometric.*

The proof of this statement is contained in [Kl1], together with a number of similar results dealing with other families of isospectral tori defined in [CS]. It would be interesting to find a proof of the conjecture of Conway and Sloane, see (2.64), using the nodal count.

As mentioned in the Introduction, in [GSS] a different nodal count was introduced for flat tori which distinguishes the tori T^+ and T^- whenever all parameters are different, as shown rigorously in [BKP]. However, it is hard to see how to define an analogue of this way of counting on general closed Riemannian manifolds.

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Appendix

See Tables 1, 2, 3 and 4.

Table 1. The vectors $m^{\pm}_{(j)}$ in terms of $l = l^+(\gamma)$ with permutation of their entries

$$
\begin{array}{ccc}
m^+_{(1)} = \begin{pmatrix} -l_1 - l_2 + l_3 \\ -l_1 + l_2 - l_4 \\ -l_1 - l_3 + l_4 \\ l_2 + l_3 + l_4 \end{pmatrix} & \xrightarrow{l_1 \mapsto -l_1} & \begin{pmatrix} l_1 - l_2 + l_3 \\ l_1 + l_2 - l_4 \\ l_1 - l_3 + l_4 \\ l_2 + l_3 + l_4 \end{pmatrix} = m^-_{(1)} \\
\Big\downarrow \begin{smallmatrix} l_1 \mapsto -l_1 \\ l_3 \mapsto -l_3 \end{smallmatrix} & & \begin{smallmatrix} l_3 \mapsto -l_3 \\ l_4 \mapsto -l_4 \end{smallmatrix} \Big\downarrow \\
m^+_{(2)} = \begin{pmatrix} l_1 - l_2 - l_3 \\ l_1 + l_2 - l_4 \\ -l_1 - l_3 - l_4 \\ -l_2 + l_3 - l_4 \end{pmatrix} & \xrightarrow{l_4 \mapsto -l_4} & \begin{pmatrix} l_1 - l_2 - l_3 \\ l_1 + l_2 + l_4 \\ -l_1 - l_3 + l_4 \\ -l_2 + l_3 + l_4 \end{pmatrix} = m^-_{(2)} \\
\Big\downarrow \begin{smallmatrix} l_3 \mapsto -l_3 \\ l_4 \mapsto -l_4 \end{smallmatrix} & & \begin{smallmatrix} l_2 \mapsto -l_2 \\ l_3 \mapsto -l_3 \end{smallmatrix} \Big\downarrow \\
m^+_{(3)} = \begin{pmatrix} l_1 - l_2 + l_3 \\ -l_1 - l_2 - l_4 \\ l_1 - l_3 - l_4 \\ -l_2 - l_3 + l_4 \end{pmatrix} & \xrightarrow{l_2 \mapsto -l_2} & \begin{pmatrix} l_1 + l_2 + l_3 \\ -l_1 + l_2 - l_4 \\ l_1 - l_3 - l_4 \\ l_2 - l_3 + l_4 \end{pmatrix} = m^-_{(3)} \\
\Big\downarrow \begin{smallmatrix} l_2 \mapsto -l_2 \\ l_4 \mapsto -l_4 \end{smallmatrix} & & \begin{smallmatrix} l_1 \mapsto -l_1 \\ l_2 \mapsto -l_2 \end{smallmatrix} \Big\downarrow \\
m^+_{(4)} = \begin{pmatrix} -l_1 - l_2 - l_3 \\ l_1 - l_2 - l_4 \\ l_1 - l_3 + l_4 \\ l_2 - l_3 - l_4 \end{pmatrix} & \xrightarrow{l_3 \mapsto -l_3} & \begin{pmatrix} -l_1 - l_2 + l_3 \\ l_1 - l_2 - l_4 \\ l_1 + l_3 + l_4 \\ l_2 + l_3 - l_4 \end{pmatrix} = m^-_{(4)}
\end{array}
$$

Table 2. The vectors $p_1^\pm$ and $p_2^\pm$ in terms of $l = l^+(\gamma)$

$$a = l_1 + l_4 \qquad A = ab = l_1^2 - l_4^2$$
$$b = l_1 - l_4 \qquad B = cd = l_2^2 - l_3^2$$
$$c = l_2 + l_3 \qquad C = \tfrac{1}{2}(ac - bd) = l_1 l_3 + l_2 l_4$$
$$d = l_2 - l_3 \qquad D = \tfrac{1}{2}(ac + bd) = l_1 l_2 + l_3 l_4$$

$$p_1^+ = \begin{pmatrix} A+B+C \\ (A+B+C)-3C-3D \\ (A+B+C)-3B-3C \\ -(A+B+C)+3A+3C \\ A+B+C-3C+3D \\ -(A+B+C) \end{pmatrix} \qquad p_1^- = \begin{pmatrix} -(A-B+D) \\ -(A-B+D)-3B+3D \\ -(A-B+D)+3C+3D \\ (A-B+D)+3C-3D \\ -(A-B+D)+3A+3D \\ A-B+D \end{pmatrix}$$

$$p_2^+ = \begin{pmatrix} -(A+B-C) \\ -(A+B-C)-3C+3D \\ -(A+B-C)+3A-3C \\ (A+B-C)-3B+3C \\ -(A+B-C)-3C-3D \\ A+B-C \end{pmatrix} \qquad p_2^- = \begin{pmatrix} -(A-B-D) \\ -(A-B-D)+3A-3D \\ -(A-B-D)+3C-3D \\ A-B-D+3C+3D \\ -(A-B-D)-3B-3D \\ A-B-D \end{pmatrix}$$

Table 3. The vectors $p_3^\pm$ and $p_4^\pm$ in terms of $l = l^+(\gamma)$

$$a = l_1 + l_2 \qquad A = ab = l_1^2 - l_2^2$$
$$b = l_1 - l_2 \qquad B = cd = l_4^2 - l_3^2$$
$$c = l_4 + l_3 \qquad C = \tfrac{1}{2}(ac - bd) = l_1 l_3 + l_2 l_4$$
$$d = l_4 - l_3 \qquad D = \tfrac{1}{2}(ac + bd) = l_1 l_4 + l_2 l_3 (=: E)$$

$$p_3^+ = \begin{pmatrix} A-B+D+3B-3D \\ A-B+D \\ A-B+D-3C-3D \\ -(A-B+D)-3C+3D \\ A-B+D \\ -(A-B+D)+3A+3D \end{pmatrix} \qquad p_3^- = \begin{pmatrix} -(A+B+C)+3C+3D \\ -(A+B+C) \\ -(A+B+C)+3B+3C \\ A+B+C-3A-3C \\ -(A+B+C) \\ A+B+C-3C+3D \end{pmatrix}$$

$$p_4^+ = \begin{pmatrix} (A-B-D)-3A+3D \\ A-B-D \\ (A-B-D)-3C+3D \\ -(A-B-D)-3C-3D \\ A-B-C \\ -(A-B-D)-3B-3D \end{pmatrix} \qquad p_4^- = \begin{pmatrix} (A+B-C)+3C-3D \\ A+B-C \\ (A+B-C)-3A+3C \\ -(A+B-C)+3B-3C \\ A+B-C \\ -(A+B-C)-3C-3D \end{pmatrix}$$

Table 4. The vectors $p_5^\pm$ and $p_6^\pm$ in terms of $l = l^+(\gamma)$

$$a = l_1 + l_3 \qquad A = ab = l_1^2 - l_3^2$$
$$b = l_1 - l_3 \qquad B = cd = l_4^2 - l_2^2$$
$$c = l_4 + l_2 \qquad C = \tfrac{1}{2}(ac - bd) = l_1 l_2 + l_3 l_4$$
$$d = l_4 - l_2 \qquad D = \tfrac{1}{2}(ac + bd) = l_1 l_4 + l_2 l_3$$

$$p_5^+ = \begin{pmatrix} (A+B+C)-3C-3D \\ (A+B+C)-3B-3C \\ A+B+C \\ -(A+B+C) \\ (A+B+C)-3A-3C \\ -(A+B+C)+3C-3D \end{pmatrix} \qquad p_5^- = \begin{pmatrix} -(A-B+D)+3B+3D \\ -(A-B+D)+3C+3D \\ -(A-B+D) \\ A-B+D \\ -(A-B+D)+3C-3D \\ (A-B+D)-3A-3D \end{pmatrix}$$

$$p_6^+ = \begin{pmatrix} -(A+B-C)-3C+3D \\ -(A+B-C)+3A-3C \\ -(A+B-C) \\ A+B-C \\ -(A+B-C)+3B-3C \\ (A+B-C)+3C+3D \end{pmatrix} \qquad p_6^- = \begin{pmatrix} -(A-B-D)+3A-3D \\ -(A-B-D)+3C-3D \\ -(A-B-D) \\ A-B-D \\ -(A-B-D)-3C-3D \\ (A-B-D)+3B+3D \end{pmatrix}$$

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