ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 14, No. 4, 2007, pp. 423-429. © Pleiades Publishing, Ltd., 2007.

On the Discrete Spectrum of Spin-Orbit Hamiltonians with Singular Interactions

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Received September 19, 2007

Abstract. We give a variational proof of the existence of infinitely many bound states placed below the continuous spectrum for spin-orbit Hamiltonians (including the Rashba and Dresselhaus cases) perturbed by measure potentials, thus extending results of J. Brüning, V. Geyler, K. Pankrashkin, J. Phys. A: Math. Theor. **40** F113–F117 (2007).

DOI: 10.1134/S1061920807040073

1. INTRODUCTION

As is well known, perturbations of the Laplacian by sufficiently localized potentials produce only finitely many negative eigenvalues, and only long-range potentials can produce an infinite discrete spectrum, see Section XII.3 in [13]. This fails if one considers perturbations of magnetic Schrödinger operators, where compactly supported perturbations demonstrate a nonclassical behavior [11, 14].

Recently, questions of this kind become of interest in the study of operators related to spintronics. Namely, as was stressed in [8], perturbations of the Rashba and Dresselhaus Hamiltonians

$$H_R = \begin{pmatrix} p^2 & \alpha(p_y + ip_x) \\ \alpha(p_y - ip_x) & p^2 \end{pmatrix}, \qquad H_D = \begin{pmatrix} p^2 & -\alpha(p_x + ip_y) \\ -\alpha(p_x - ip_y) & p^2 \end{pmatrix}$$

(where α stands for a constant expressing the strength of the spin-orbit coupling [6, 12, 16]) by localized spherically symmetric negative potentials produce infinitely many eigenvalues below the continuous spectrum; in the proof, some approximations have been used. In the paper [4], we gave a rigorous proof of this effect for rather general negative potentials without any symmetry conditions. In the present note, we extend these results and obtain similar estimates for operators of the form $H = H_0 + \nu$, where H_0 is an unperturbed spin-orbit Hamiltonian and ν is a measure (whose support can have Lebesgue measure zero). In particular, for the Rashba and Dresselhaus Hamiltonians, we show that negative perturbations supported by curves always produce infinitely many bound states below the threshold.

2. DEFINITION OF HAMILTONIANS

Denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ of two-dimensional spinors; by $\mathcal{F}$ we denote the Fourier transform $\mathcal{F}\colon\ L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$; in this case, $\mathcal{F}_2 := \mathcal{F} \otimes 1$ is the Fourier transform in $\mathcal{H}$. Let H_0 be the self-adjoint operator in $\mathcal{H}$ whose Fourier transform $\widehat{H}_0 := \mathcal{F}_2 H_0 \mathcal{F}_2^{-1}$ is the multiplication by the matrix

$$\widehat{H}_0(p) = \begin{pmatrix} p^2 & A(p) \\ \overline{A(p)} & p^2 \end{pmatrix}, \qquad p \in \mathbb{R}^2, \tag{1}$$

where A is a continuous complex function on $\mathbb{R}^2$. Assume that

$$\limsup_{p\to\infty} |A(p)| p^{-2} < 1. \tag{2}$$

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Clearly, H_0 has no discrete spectrum; its spectrum is the union of images of two functions $\lambda_\pm$ (dispersion laws): $\lambda_\pm(p) = p^2 \pm |A(p)|$, and hence $\operatorname{spec} H_0 = [\kappa, +\infty)$, where $\kappa := \inf\{p^2 - |A(p)| : p \in \mathbb{R}^2\} > -\infty$. Moreover, there holds

$$M(p)\widehat{H}_0(p)M^*(p) = \begin{pmatrix} \lambda_+(p) & 0 \\ 0 & \lambda_-(p) \end{pmatrix}, \qquad M(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\arg A(p)} & 1 \\ e^{-i\arg A(p)} & -1 \end{pmatrix}, \qquad p \in \mathbb{R}^2. \tag{3}$$

Write $S := \{p \in \mathbb{R}^2 : \lambda_-(p) = \kappa\}$; this is a nonempty compact set. Assume that

$$\text{the function } |A(p)| \text{ is of class } C^2 \text{ in a neighborhood of } S. \tag{4}$$

For the Rashba and Dresselhaus Hamiltonians, one has $\kappa = -\alpha^2/4$, S is the circle $\{p : 2|p| = |\alpha|\}$, and condition (4) obviously holds.

The two conditions (2) and (4) ensure that, for any $p_0 \in S$, there is a constant $c(p_0) > 0$ for which

$$0 \leqslant \lambda_-(p) - \kappa \leqslant c(p_0)|p - p_0|^2 \quad \text{for all} \quad p \in \mathbb{R}^2. \tag{5}$$

Choose a positive Radon measure m on $\mathbb{R}^2$ and a bounded Borel measurable function $h\colon \mathbb{R}^2 \to \mathbb{R}$ such that there are constants $a \in (0,1)$ and $b > 0$ for which

$$\int_{\mathbb{R}^2} (1 + |h(x)|^2)|f(x)|^2 m(dx) \leqslant a \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx + b \int_{\mathbb{R}^2} |f(x)|^2 dx \tag{6}$$

for any f in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Write $\nu := hm$. Assumption (6) is satisfied for any (bounded) h if m belongs to the Kato class of measures, i.e.,

$$\lim_{\varepsilon\to 0+} \sup_{x\in\mathbb{R}^2} \int_{|x-y|<\varepsilon} \log(1/|x-y|)\big|m\big|(dy) = 0.$$

This holds, for example, for δ-type measures concentrated on C^1 curves under some regularity conditions (these conditions are satisfied for compact curves and straight lines); for details, see Section 4 in [3].

Our aim now is to give a rigorous definition of the operator given by the formal expression

$$H = H_0 + \nu. \tag{7}$$

Since $\mathcal{S}(\mathbb{R}^2)$ is dense in the Sobolev space $H^1(\mathbb{R}^2)$, there is a unique linear bounded transformation J defined by the rule $J\colon H^1(\mathbb{R}^2) \to L^2(\mathbb{R}^2, m)$, $Jf = f$ for all $f \in \mathcal{S}(\mathbb{R}^2)$. Set $J_2 := J \otimes 1$; this is an operator acting from $H^1(\mathbb{R}^2) \otimes \mathbb{C}^2$ to $L^2(\mathbb{R}^2, m) \otimes \mathbb{C}^2$. For a continuous function f, denote the corresponding equivalence classes in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}, m)$ by the same letter f.

We note now that the operator H_0 can be represented as

$$H_0 = -\Delta_2 + L, \qquad -\Delta_2 := -\Delta \otimes 1, \qquad \widehat{L} := \mathcal{F}_2 L \mathcal{F}_2^{-1} = \begin{pmatrix} 0 & A(p) \\ \overline{A(p)} & 0 \end{pmatrix}.$$

(Here Δ stands for the scalar two-dimensional Laplacian.)

Lemma 1. *The operator L is relatively bounded with respect to Δ_2, and $\|L(-\Delta_2 + \lambda)^{-1}\| < 1$ for $\lambda \to +\infty$.*

Proof. The relative boundedness is obvious. Thus, let us prove only the norm estimate. Passing to the Fourier transform, we must show that

$$\sup_{p\in\mathbb{R}^2} \big|A(p)/(p^2 + \lambda)\big| < 1, \qquad \lambda \to +\infty. \tag{8}$$

By (2), there are numbers $a < 1$ and $R > 0$ such that $|A(p)|/p^2 \leqslant a$ for all p with $|p| > R$. Then we obviously have

$$\big|A(p)/(p^2 + \lambda)\big| < a, \qquad |p| > R, \quad \lambda > 0. \tag{9}$$

Due to the continuity of A, there is a $C > 0$ with $|A(p)| \leqslant C$ for $|p| \leqslant R$. Then. obviously, there is a $\lambda_0 > 0$ such that

$$\big|A(p)/(p^2 + \lambda)\big| \leqslant C\lambda^{-1} < a, \qquad |p| \leqslant R, \quad \lambda > \lambda_0. \tag{10}$$

Combining (9) with (10), we arrive at (8).

Equation (6) and Lemma 1 imply that, by the KLMN theorem, the quadratic form $q(f,g) = q_0(f,g) + \nu(f,g)$, where

$$\nu(f,g) := \langle hJ_2 f, J_2 g\rangle_{L^2(\mathbb{R}^2)\otimes\mathbb{C}^2} \equiv \int_{\mathbb{R}^2} \langle J_2 f(x), J_2 g(x)\rangle_{\mathbb{C}^2}\,\nu(dx), \qquad \nu(dx) = h(x)m(dx),$$

and q_0 is the quadratic form associated with H_0, is semibounded below and closed on $H^1(\mathbb{R}^2)\otimes\mathbb{C}^2$ and, hence defines a certain self-adjoint operator H semibounded below. If the measure ν is absolutely continuous with respect to the Lebesgue measure, i.e., $\nu(dx) = V(x)dx$ for some locally integrable function V, then the above procedure gives the usual sum of the form $H = H_0 + V$, and thus one preserves the same notation for the general case, $H_0 + \nu := H$.

Repeating the procedure of Section 2 in [3], one can express the resolvent of H in terms of the resolvent of H_0. Combining this with the explicit expressions for the Green function for H_0 [5], one can obtain rather detailed formulas for the Green function of H, but we do not need them below. The following assertion about the spectral properties of H is important for us.

Theorem 2. *If the measure m is finite, then the essential spectra of H_0 and H coincide.*

Proof. The paper [2] deals with a rather detailed spectral analysis of operators defined by sums of quadratic forms. According to Theorem 7 in [2], we must prove only that the operator $J_2(H_0 - z)^{-1} : L^2(\mathbb{R}^2)\otimes\mathbb{C}^2 \to L^2(\mathbb{R}^2, m)\otimes\mathbb{C}^2$ is compact. Note that, for sufficiently large λ, the operator $1 + L(-\Delta_2 + \lambda)^{-1}$ has bounded inverse defined everywhere, due to Lemma 1. Hence, $(H_0 + \lambda)^{-1} = (-\Delta_2 + \lambda)^{-1}\big(1 + L(-\Delta_2 + \lambda)^{-1}\big)^{-1}$, and the compactness of $J_2(H_0 + \lambda)^{-1}$ would follow from the compactness of $J(-\Delta + \lambda)^{-1} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2, m)$ because $J_2(-\Delta_2 + \lambda)^{-1} \equiv \big(J(-\Delta + \lambda)^{-1}\big)\otimes 1$. At the same time, $J(-\Delta + \lambda)^{-1} = B^*$,

$$B\colon L^2(\mathbb{R}^2, m) \to L^2(\mathbb{R}^2), \qquad Bf(x) = \int_{\mathbb{R}^2} G_0(x, y; \lambda) f(y) m(dy) \quad \text{a.e.},$$

where G_0 stands for the Green function of the two-dimensional Laplacian. As was proved in Lemma 2.3 of [3], B is compact. Therefore, $J(-\Delta + \lambda)^{-1}$ is also compact, and the theorem is proved.

From now on, we assume that the measure m is finite.

3. GENERAL PERTURBATIONS

Below, for a distribution f, denote the Fourier transform of f by $\widehat{f}$. A Hermitian $n\times n$ matrix C is said to be *positive definite* if $\langle\xi, C\xi\rangle > 0$ for any nonzero $\xi\in\mathbb{C}^n$ and *positive semi-definite* if the above inequality is nonstrict. One similarly introduces *negative definite* and *negative semi-definite* matrices.

The following result differs only in minor details from the main result in [5].

Theorem 3. *Let $N\in\mathbb{N}$; assume that the Fourier transform $\widehat{\nu}$ satisfies the following condition: there are N points $p_1,\dots,p_N\in S$ such that the matrix $\big(\widehat{\nu}(p_j - p_k)\big)_{j,k=1}^N$ is negative definite. Then H has at least N eigenvalues below κ (counted according to their multiplicities).*

Proof. According to the max-min principle, it is sufficient to show that we can find N vectors $\Psi_m\in\mathcal{H}$, $m = 1,\dots,N$, such that the matrix with the entries $(q-\kappa)(\Psi_j,\Psi_k)$, $j,k = 1,\dots,N$, $(q-\kappa)(\Phi,\Psi) := q(\Phi,\Psi) - \kappa\langle\Phi,\Psi\rangle$ is negative definite.

Set $f_a(x) := \exp\big(-|x|^a/2\big)$, $x\in\mathbb{R}^2$, with $a > 0$. Clearly, $f_a\in H^1(\mathbb{R}^2)$. It is observed in [17] that

$$\int_{\mathbb{R}^2} \big|\nabla f_a(x)\big|^2\,dx = (\pi/2)a. \tag{11}$$

Further, by the Lebesgue dominated convergence theorem, we have

$$\lim_{a\to 0+} \int_{\mathbb{R}^2} |f_a(x)|^2\,\nu(dx) = e^{-1}\int_{\mathbb{R}^2}\nu(dx).$$

Let $\widehat{f_a}$ be the Fourier transform of f_a. Take spinors Ψ_j such that their Fourier transforms $\widehat{\Psi}_j$ are of the form $\widehat{\Psi}_j(p) = M^*(p)\psi_j(p)$, where

$$\psi_j(p) = \begin{pmatrix} 0 \\ \widehat{f}_a(p - p_j) \end{pmatrix} \tag{12}$$

and $M(p)$ is taken from (3). We show that, if a is sufficiently small, then the matrix $(q-\kappa)(\Psi_j, \Psi_k)$ is negative definite. To this end, it suffices to show that

$$\lim_{a\to 0}(q_0 - \kappa)(\Psi_j, \Psi_k) = 0, \tag{13}$$

$$\lim_{a\to 0}\nu(\Psi_j, \Psi_k) = 2\pi e^{-1}\widehat{\nu}(p_j - p_k) + \pi e^{-1}\exp(i\arg A(p_k) - i\arg A(p_j))\widehat{\nu}(p_j - p_k) \tag{14}$$

for all j and k.

By the definition of Ψ_j, one has

$$\begin{aligned}\bigl|(q_0 - \kappa)(\Psi_j, \Psi_k)\bigr| &= \Bigl|\int_{\mathbb{R}^2}(\lambda_-(p) - \kappa)\overline{\widehat{f}_a(p - p_j)}f_a(p - p_k)\,dp\Bigr| \\ &\leqslant \sqrt{\int_{\mathbb{R}^2}\bigl(\lambda_-(p) - \kappa\bigr)\bigl|\widehat{f}_a(p - p_j)\bigr|^2\,dp}\,\sqrt{\int_{\mathbb{R}^2}\bigl(\lambda_-(p) - \kappa\bigr)\bigl|\widehat{f}_a(p - p_k)\bigr|^2\,dp}.\end{aligned}$$

On the other hand, by (5) and (11), one has

$$\begin{aligned}0 \leqslant \int_{\mathbb{R}^2}\bigl(\lambda_-(p) - \kappa\bigr)\bigl|\widehat{f}_a(p - p_j)\bigr|^2\,dp &\leqslant c(p_j)\int_{\mathbb{R}^2}(p - p_j)^2\bigl|\widehat{f}_a(p - p_j)\bigr|^2\,dp \\ &= c(p_j)\int_{\mathbb{R}^2}p^2\bigl|\widehat{f}_a(p)\bigr|^2\,dp = c(p_j)\int_{\mathbb{R}^2}\bigl|\nabla f_a(x)\bigr|^2\,dx = \frac{\pi}{2}c(p_j)a,\end{aligned}$$

which proves (13). As for (14), one has

$$\begin{aligned}\nu(\Psi_j, \Psi_k) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{\nu}(p - q)\bigl\langle\widehat{\Psi}_j(p), \widehat{\Psi}_k(q)\bigr\rangle_{\mathbb{C}^2}dp\,dq \\ &= \frac{2 + \exp(i\arg A(p_k) - i\arg A(p_j))}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{\nu}(p - q)\bigl\langle\psi_j(p), \psi_k(q)\bigr\rangle_{\mathbb{C}^2}dp\,dq.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{\nu}(p - q)\bigl\langle\psi_j(p), \psi_k(q)\bigr\rangle_{\mathbb{C}^2}dp\,dq &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\widehat{\nu}(p - q)\overline{\widehat{f}_a(p - p_j)}\widehat{f}_a(q - p_k)\,dp\,dq \\ &= \int_{\mathbb{R}^2}e^{i\langle p_j - p_k, x\rangle}\bigl|f_a(x)\bigr|^2\,\nu(dx) \ \overset{a\to 0}{\longrightarrow}\ 2\pi e^{-1}\widehat{\nu}(p_j - p_k),\end{aligned}$$

and the theorem is proved.

The above theorem enables us to generalize some classical results. For example, the condition $\int_{\mathbb{R}^2}\nu(dx) \equiv 2\pi\widehat{\nu}(0) < 0$ ensures that H has at least one eigenvalue below κ.

To formulate further corollaries, denote by $\#S$ the number of points in S if S is finite and ∞ otherwise.

Corollary 4. *Let $\nu \leqslant 0$ and let $\operatorname{supp}\nu$ have positive Lebesgue measure. Then H has at least $\#S$ eigenvalues below κ.*

Proof. We show that the matrix $\bigl(\widehat{\nu}(p_j - p_k)\bigr)_{j,k=1}^N$ is negative definite for any choice of pairwise different points $p_1, \dots, p_N \in \mathbb{R}^2$. By the Bochner theorem,

$$-\sum_{j,k}\widehat{\nu}(p_j - p_k)\bar{\xi}_j\xi_k \geqslant 0 \quad \text{for any} \quad (\xi_j) \in \mathbb{C}^N,$$

and it remains to note that

$$\sum_{j,k}\widehat{\nu}(p_j - p_k)\bar{\xi}_m\xi_n \neq 0 \quad \text{for} \quad (\xi_j) \neq 0.$$

In fact,

$$\text{if} \quad \sum_{j,k}\widehat{\nu}(p_j - p_k)\bar{\xi}_j\xi_k = 0, \quad \text{then} \quad \int_{\mathbb{R}^2}\Bigl|\sum_j \xi_j e^{i\langle p_j, x\rangle}\Bigr|^2\,\nu(dx) = 0\,;$$

therefore, $\sum_j \xi_j e^{i\langle p_j, x\rangle} = 0$ on the support of ν. As the exponents $e^{i\langle p_j, x\rangle}$ are real analytic, and $\sum_j \xi_j e^{i\langle p_j, x\rangle} = 0$ on a set of positive Lebesgue measure, the equality $\sum_j \xi_j e^{i\langle p_j, x\rangle} = 0$ is valid everywhere on $\mathbb{R}^2$. On the other hand, the functions $e^{i\langle p_j, x\rangle}$ are linearly independent, and we obtain $\xi_j = 0$ for all j.

Corollary 4 shows the existence of infinitely many eigenvalues below κ for perturbations of the Rashba and Dresselhaus Hamiltonians by negative measures with support of positive Lebesgue measure, i.e., given by negative regular potentials, sums of a negative regular potential and a negative δ-measure supported by a curve, etc. The question of whether or not Corollary 4 remains valid if the support of m has zero Lebesgue measure is of interest; the above arguments do not work in that case.

In [7], we have found an interesting condition permitting to handle a class of singular perturbations without assumptions on the Lebesgue measure of the support.

Corollary 5. *Let $\nu \leqslant 0$ and let the intersection of* $\operatorname{supp}\nu$ *with some circle be an infinite set. Then H has at least $\#S$ eigenvalues below κ. In particular, this holds if m is spherically symmetric and $h < 0$.*

Proof. As was proved in [7], under the above assumptions, the matrix $\big(\widehat{\nu}(p_j - p_k)\big)$ is negative definite for any choice of pairwise different numbers p_j.

4. PERTURBATIONS SUPPORTED BY CURVES

The above results enable one to study perturbations supported by curves in rather particular symmetric cases only. In this section, we consider perturbations supported by curves in greater detail.

Lemma 6. *Assume that*

(a) *S contains a C^1 arc which is not an interval*;
(b) *m is compactly supported*;
(c) *$h(x) < 0$ for a.e. x with respect to the measure m*;
(d) *the Fourier transform $\hat{\nu} \equiv \widehat{hm}$ vanishes at infinity, at least along some straight line, i.e., $\hat{\nu}(r\cos\alpha, r\sin\alpha) \to 0$ as $r \to \pm\infty$ for some $\alpha \in [0, 2\pi)$.*

In this case, the assumptions of Theorem 3 *are satisfied for any N.*

We first prove a simple lemma.

Lemma 7. *Let C be a C^1 arc in $\mathbb{R}^2$ which differs from any interval. In this case, the set $X = \{p - q : p, q \in C\} \subset \mathbb{R}^2$ has an interior point.*

Proof. Let $t \mapsto \big(x(t), y(t)\big)$, $t \in [0,1]$, be a regular parametrization of C. Consider the mapping $f : (s,t) \mapsto \big(x(t) - x(s), y(t) - y(s)\big)$, $s, t \in [0,1]$. Clearly, $X = f\big([0,1] \times [0,1]\big)$. Since C is not an interval, the Jacobian

$$\det \begin{pmatrix} -x'(s) & -y'(s) \\ x'(t) & y'(t) \end{pmatrix}$$

does not vanish at some point $(s_0, t_0) \in (0,1) \times (0,1)$. By the implicit function theorem, f is invertible near (s_0, t_0), and therefore the image of some neighborhood of (s_0, t_0) is an open set.

Proof of Lemma 6. Assumption (b) implies the analyticity of $\hat{\nu}$. We claim that, for any N, there are points $p_j \in S$, $j = 1, \dots, N$, such that the matrix $\mathcal{V}(p_1, \dots, p_n)$ with the entries $\mathcal{V}_{jk} = \hat{\nu}(p_j - p_k)$ is negative definite. By the Bochner theorem, $\mathcal{V}$ is negative semi-definite for any choice of p_j, and it remains to show that $\det \mathcal{V}(p_1, \dots, p_N) \neq 0$ for some choice of points p_j in S.

Now we proceed by induction on N. For $N = 1$, one has $\det \mathcal{V} = \hat{\nu}(0) < 0$ by (c).

Let $N \geqslant 2$. Assume that there are $p'_j \in S$, $j = 1, \dots, N-1$, such that $\det \mathcal{V}(p'_1, \dots, p'_{N-1}) \neq 0$.

Note that $\mathcal{V}(p_1,\dots,p_N)$ is a function of $N-1$ variables q_j, $q_j := p_j - p_N$, $j = 1,\dots,N-1$, $q_j \in X$, $X = \{p-q : p,q \in S\}$, $\mathcal{V}(p_1,\dots,p_N) = \widetilde{\mathcal{V}}(q_1,\dots,q_{N-1})$, and $\mathcal{V}$ is analytic in q_j. Hence, if $\det\mathcal{V}(p_1,\dots,p_N) = 0$ for any choice of $p_j \in S$, then, by Lemma 7, $\det\widetilde{\mathcal{V}}(q_1,\dots,q_{N-1})$ vanishes on an open set and, due to the analyticity, $\det\mathcal{V}(p_1,\dots,p_N) = 0$ for any N-tuple $(p_1,\dots,p_N)$ in $(\mathbb{R}^2)^N$.

On the other hand, by (c), we have $\mathcal{V}_{jj} = \hat\nu(0) < 0$ independently of $(p_1,\dots,p_N)$ for every j. Using the determinant expansion and condition (d), we obtain

$$\det\mathcal{V}(p_1',\dots,p_{N-1}',p_N) \to \hat\nu(0)\det\mathcal{V}(p_1',\dots,p_N') \neq 0 \quad \text{as} \quad p_N = r(\cos\alpha,\sin\alpha) \quad \text{and} \quad r \to \pm\infty.$$

This contradiction completes the proof.

Item (d) in Lemma 6 is certainly the finest one among the assumptions, and there is a vast literature discussing the decay rates of the Fourier transforms of measures, see, e.g., [1, 9, 10]. For our purposes, it suffices to consider the following two cases.

Lemma 8. *Let Γ be a compact C^2 curve with nonvanishing curvature and $\nu = \delta_\Gamma$. Then assumption (d) in Lemma 6 is satisfied.*

Proof. Let $[0,l] \ni t \mapsto \big(x(t),y(t)\big)$ be the natural parametrization of Γ. As the curvature does not vanish, one has

$$\big|x'(t)y''(t) - x''(t)y'(t)\big| \geqslant c > 0 \quad \text{for all} \quad t. \tag{15}$$

Consider the expression

$$\hat\nu(p) = \frac{1}{2\pi}\int_0^l \exp\Big[-ir\big(\cos\alpha x(t) + \sin\alpha y(t)\big)\Big]dt, \qquad p = r(\cos\alpha,\sin\alpha).$$

By (15), there is a $\delta > 0$ for which one can divide Γ into several smooth pieces such that one of the following conditions is satisfied on each of these pieces:

(1) $\big|\cos\alpha x'(t) + \cos\alpha y'(t)\big| \geqslant \delta$,

(2) $\big|\cos\alpha x''(t) + \cos\alpha y''(t)\big| \geqslant \delta$.

Hence, by the van der Corput lemma on oscillatory integrals, [15, Sec. VIII.1, Prop. 2], one has $\hat\nu(p) \to 0$ for $r \to \infty$.

Lemma 9. *Let Γ be a line segment and $\nu = \delta_\Gamma$. Then assumption (d) in Lemma 6 is satisfied.*

Proof. Let $[0,1] \ni t \mapsto (x_0 + at, y_0 + bt)$ be a parametrization of Γ. Writing

$$\begin{aligned}\hat\nu(p) &= \frac{\sqrt{a^2+b^2}}{2\pi}\int_0^1 \exp\Big[-ir\big(\cos\alpha(x_0+at) + \cos\alpha(y_0+bt)\big)\Big]\,dt \\ &= \frac{\sqrt{a^2+b^2}\,\exp\big(-ir(x_0\cos\alpha + y_0\sin\alpha)\big)}{2\pi}\int_0^1 \exp\Big[-irt\big(a\cos\alpha + b\sin\alpha\big)\Big]\,dt, \\ &\qquad p = r(\cos\alpha,\sin\alpha),\end{aligned}$$

we immediately see that $\hat\nu(r\cos\alpha, r\sin\alpha) \to 0$ as $r \to \pm\infty$ for any α with $a\cos\alpha + b\sin\alpha \neq 0$ by the Riemann–Lebesgue theorem.

Now we can use Lemmas 6, 8, and 9 to prove a general result on singular interactions supported by curves.

Theorem 10. *Assume that S contains a C^1 arc which is not a line segment. Let Γ be a C^2 curve, let $m = \delta_\Gamma$, and let the restriction of h to Γ be a negative continuous function. In this case, the operator H has infinitely many eigenvalues below the essential spectrum.*

Proof. There exists a part of Γ, Γ', with the following properties:

(1) Γ' is a compact C^2 curve;

(2) Γ' either has nonvanishing curvature or is a line segment;

(3) $h|_{\Gamma'} \leqslant -c$, $c > 0$.

Write $\nu = \nu_1 + \nu_2$, $\nu_1 = -c\delta_{\Gamma'}$, $\nu_2 := \nu - \nu_1$. By construction, $\nu_2 \leqslant 0$. By Lemma 6, for any N, there are points $p_1, \dots, p_N \in S$ such that the matrix $\big(\hat\nu_1(p_j - p_k)\big)$ is negative definite. Since ν_2 is nonpositive, the matrix $\big(\hat\nu_2(p_j - p_k)\big)$ is at least negative semi-definite by the Bochner theorem, and $\big(\hat\nu(p_j - p_k)\big)$ is thus negative definite. Thus, H has infinitely many eigenvalues below the essential spectrum by Theorem 3.

Assumption (a) of Lemma 6 obviously holds for the Rashba and Dresselhaus Hamiltonians. Hence, Theorem 10 guarantees the existence of infinitely many eigenvalues below the continuous spectrum under negative perturbations supported by smooth curves.

If the set S for the unperturbed Hamiltonian H_0 is very "bad" and contains no smooth arc, then an analog of Lemma 7 becomes a difficult problem in general topology. Nevertheless, the assumptions of Lemma 7 are naturally satisfied in reasonable examples, including the above Rashba and Dresselhaus Hamiltonians.

We note in conclusion that the method proposed to estimate the number of eigenvalues is quite universal but very rough; it does not take into account, e.g., the Kramers degeneracy (all eigenvalues of perturbed Rashba Hamiltonians must be at least twice degenerate).

ACKNOWLEDGMENTS

The work was supported by the Deutsche Forschungsgemeinschaft. We thank Johannes Brasche, Georgi Raikov, and Grigori Rozenblum for useful discussions and several bibliographical hints.

REFERENCES

1. L. Brandolini, G. Gigante, A. Greenleaf, A. Iosevich, A. Seeger, and G. Travaglini, "Average Decay Estimates for Fourier Transforms of Measures Supported on Curves," J. Geom. Anal. **17** 15–40 (2007).
2. J. F. Brasche, "Upper Bounds for Neumann–Schatten Norms," Potential Anal. **14**, 175–205 (2001).
3. J. F. Brasche, P. Exner, Yu. A. Kuperin, and P. Šeba, "Schrödinger Operators with Singular Interactions," J. Math. Anal. Appl. **184**, 112–139 (1994).
4. J. Brüning, V. Geyler, and K. Pankrashkin, "On the Number of Bound States for Weak Perturbations of Spin-Orbit Hamiltonians," J. Phys. A: Math. Theor. **40**, F113–F117 (2007).
5. J. Brüning, V. Geyler, and K. Pankrashkin, "Explicit Green Functions for Spin-Orbit Hamiltonians," J. Phys. A: Math. Theor. **40**, F697–F704 (2007).
6. Yu. A. Bychkov and E. I. Rashba, "Properties of a 2D Electron Gas with Lifted Spectral Degeneracy," JETP Lett. **39**, 78–81 (1984).
7. W. zu Castell, F. Filbir, and R. Szwarc, "Strictly Positive Definite Functions in $\mathbb{R}^d$," J. Approx. Theory **137**, 277–280 (2005).
8. A. V. Chaplik and L. I. Magarill, "Bound States in a Two-Dimensional Short Range Potential Induced by Spin-Orbit Interaction," Phys. Rev. Lett. **96**, 126402 (2006).
9. W. Littman, "Fourier Transforms of Surface-Carried Measures and Differentiability of Surface Averages," Bull. Amer. Math. Soc. **69**, 766–770 (1963).
10. B. P. Marshall, "Decay Rates of Fourier Transforms of Curves," Trans. Amer. Math. Soc. **310**, 115–126 (1988).
11. G. D. Raikov and S. Warzel, "Quasi-Classical Versus Non-Classical Spectral Asymptotics for Magnetic Schrödinger Operators with Decreasing Electric Potentials," Rev. Math. Phys. **14**, 1051–1072 (2002).
12. E. I. Rashba and V. I. Sheka, "Electric-Dipole Spin Resonance," in G. Landwehr and E. I. Rashba, Eds., *Landau Level Spectroscopy* (Elsevier, Amsterdam, 1991), pp. 131–206.
13. M. Reed and B. Simon, *Methods of Modern Mathematical Physics,* Vol IV: *Analysis of Operators* (Academic Press, New York–London, 1978).
14. G. Rozenblum and G. Tashchiyan, "On the Spectral Properties of the Perturbed Landau Hamiltonian," preprint `math-ph/0605038` in `arXiv`.
15. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* (Princeton University Press, Princeton, 1993).
16. R. Winkler, *Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Springer, Berlin, 2003).
17. K. Yang and M. de Llano, "Simple Variational Proof That Any Two-Dimensional Potential Well Supports at Least One Bound State," Amer. J. Phys. **57**, 85–86 (1989).