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# Approximate Formulas for Eigenvalues of The Laplace Operator on a Torus Arising in Linear Problems with Oscillating Coefficients

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**Abstract.** In the paper, using relatively simple formulas derived in the abstract perturbation theory of selfadjoint operators, we obtain explicit asymptotic formulas for a family of elliptic operators of Laplace type that arise in linear problems with rapidly oscillating coefficients.

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*To the blessed memory of*
*Marko Iosifovich Vishik (19.X.1921 - 23.VI.2012),*
*an outstanding scientist and a great man.*

## 1. INTRODUCTION

Let $n$ and $m$ be positive integers, $n \leqslant m$, and $C(y,x) > 0$ and $V(y,x)$ be smooth real functions defined on $\mathbb{T} \times \mathbb{R}^n$, where $\mathbb{T}$ stands for the $n$-dimensional torus with the coordinates $y = (y_1, \dots, y_n), y_j \in [2\pi]$ and $x = (x_1, x_2, \dots, x_m)$, i.e., $C$ and $V$ are $2\pi$-periodic with respect to each of the variables $y_1, \dots, y_n$. Further, let $m$ smooth functions $\Theta_j$ (phases) be given; it is assumed that the phases $\Theta_j$ are locally not collinear, i.e., that the rank of the matrix $\Theta_x$ of the rows. is equal to $n$ for any $x$. As was proved in [1, 2] (see also [3, 4, 5, 6]), when constructing rapidly varying asymptotic solutions of equations with rapidly oscillating coefficients

$$i\frac{\partial \psi}{\partial t} = (-\langle \nabla, C^2(\frac{\Theta(x)}{\mu}, x)\nabla\rangle + V)\psi \quad \text{and} \quad \frac{\partial^2 \psi}{\partial t^2} = -(-\langle \nabla, C^2(\frac{\Theta(x)}{\mu}, x)\nabla\rangle + V)\psi,$$

where $\mu$ stands for a small positive parameter, the following problem arises: to construct eigenfunctions and eigenvalues of a family of elliptic operators on the torus $\mathbb{T}$ which depend on $m$-dimensional parameters (column vectors) $p = (p_1, \dots, p_m)^T \in \mathbb{R}^m$ and $x = (x_1, \dots, x_m)^T \in \mathcal{K}$, where $\mathcal{K}$ stands for some compactum in $\mathbb{R}^m$. This family of operators is defined as follows. Introduce the following notation: denote by $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^m$, by $\nabla_y = \frac{\partial}{\partial y}$ the vector (column)-operator of the gradient in $\mathbb{R}^n_y$, and by $\nabla^\theta_y$ the vector-operator of the skew gradient $\nabla^\theta_y = \Theta_x(x)\nabla_y$. Further, write

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$\triangle_y^\Theta = \langle \nabla_y^\Theta, C^2(y,x)\nabla_y^\Theta\rangle$ and $D = \langle p, \nabla_y^\Theta\rangle$. Introduce the space $L_2(\mathbb{T})$ with "normalized" inner product and the norm

$$(\phi_1,\phi_2) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}} \overline{\phi_1(y)}\phi_2(y)dy^2, \quad \|\phi\| = \sqrt{(\phi,\phi)}, \tag{1.1}$$

where the bar stands for complex conjugation, and the average over the torus is

$$\langle w\rangle_{\mathbb{T}} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} w(y)dy_1dy_2. \tag{1.2}$$

Then the operators we are interested in and the corresponding spectral problems in the space related spectral problems in the space $L_2(\mathbb{T})$ are of the form[1]

$$\mathcal{H} = \langle (p - i\nabla_y^\Theta), C^2(y,x)(p - i\nabla_y^\Theta)\rangle = -\triangle_y^\Theta - i(DC^2(y,x) + C^2(y,x)D) + C^2(y,x)p^2, \tag{1.3}$$

$$\mathcal{H}\chi(y,x,p) = L(x,p)\chi(y,x,p),\ \chi\in L_2(\mathbb{T}),\ \|\chi(y,x,p)\| = 1. \tag{1.4}$$

Note that, setting $\chi = \exp(i(\Theta_x^*\Theta_x)^{-1}\langle\Theta_x^* p, y\rangle)\chi_b$, one can reduce the latter problem to that for Bloch functions $\chi_b$ for the Laplace operator $-\triangle_y^\theta$.

It follows from the general theory of elliptic operators on compact manifolds that $\mathcal{H}$ with the domain $C^\infty(\mathbb{T})$ is essentially selfadjoint in $L_2(\mathbb{T})$ and has a complete system of eigenfunctions $\chi^k$, $k = 0, 1, \dots$, with real eigenvalues $L^k(x,p)$ (see, e.g., [7]). Note that, for $p = 0$, the operator $\mathcal{H}$ becomes $-\triangle_y^\Theta$, which is an elliptic operator on the torus $\mathbb{T}$, and its minimum eigenvalue and the corresponding normalized eigenfunction can readily be found and are well known,

$$L^0(x,0) = 0, \qquad \chi^0(y,x,0) = 1, \tag{1.5}$$

and here the eigenvalue $L^0(x,0) = 0$ is nondegenerate, and hence is separated from the other eigenvalues. These assertions are established in the standard way by using the "energy equation"

$$(-\triangle_y^\Theta u, u)_{L^2(\mathbb{T})} = (C^2\nabla_y^\Theta u, \nabla_y^\Theta u)_{L^2(\mathbb{T})}.$$

*Since, everywhere below, we restrict ourselves to the minimal eigenvalue and to the corresponding eigenfunction, in order to avoid cumbersome notation, we omit the superscript* 0 *in our notation and write*

$$L^0 = L, \qquad \chi^0 = \chi.$$

As was proved in [8], for $x$ in a compact set $K$ in $\mathbb{R}^3$ and for $p$ sufficiently small, the eigenvalue $L$ of the operator $\mathcal{H}$ is nondegenerate and analytic in $p$, and the function $\chi(y,x,p)$ can be chosen to be smooth with respect to $(x,p)$ and analytic in $p$ (for a simple proof of these assertions, see also [9]).

The problem is to find $\chi$ and $\mathcal{E}$. Represent the function $C$ in the form

$$C^2 = C_0^2 + \tilde{a}(y,x), \quad C_0^2 = \langle C^2\rangle_{\mathbb{T}}. \tag{1.6}$$

For a variable function $C^2$, it is practically impossible to evaluate $\chi$ and $L$ explicitly, and one may speak only of the perturbation theory with respect to the parameters $p$ in a neighborhood of the point $p = 0$ or under the assumption that $\tilde{a}(y)$ is small, i.e., under the assumption that $\tilde{a} = \delta a(y)$, where $\delta$ is a small positive parameter. For applications, it is of interest to have formulas for the expansion of $L$ in the parameters (variables) $p$ up to order four inclusive. The objective of this paper is to obtain formulas of high-order perturbation theory with respect to $p$ and/or $\delta$.Certainly, the way to derive these formulas in well known in operator theory. However, we are interested in explicit formulas and in a specific case.

The paper is organized as follows. In Section 2 we present formulas of abstract perturbation theory for a selfadjoint operator of the form $A+\varepsilon B$, and apply these formulas to obtain an expansion of $L$ with accuracy up to $|p|^5$ in Section 3 (Proposition 1) and realize these expansions in the single-phase case in Sec. 4. Formulas for $L$ in the multi-phase case, which are obtained with accuracy up to $O(\delta^3)$, are presented in Section 4 (Proposition 2).

[1]We practically follow the notation introduced in [1]; however, only to simplify this notation, in this paper we omit the subscript "0" at the symbols $L$, $\chi$, and $\mathcal{H}$.

## 2. HIGH-ORDER FORMULAS OF ABSTRACT PERTURBATION THEORY

Let us first recall some useful formulas of abstract perturbation theory which we could not find in the literature. Introduce a fictitious parameter $\varepsilon$ (which has the meaning of $|p|$ or $\delta$), represent the problem of zero-order approximation in the form

$$(A+\varepsilon B)\varphi=\lambda\varphi, \quad \|\varphi\|=1, \tag{2.1}$$

and assume that $\lambda=\lambda(\varepsilon)$ is a multiplicity-free eigenvalue for sufficiently small values of $\varepsilon$. The selfadjoint operators $A$ and $B$ acting on some Hilbert space $\mathbf{H}$ with inner product $(\cdot\,,\cdot)$ can also depend on the parameter $\varepsilon$, but only regularly, and we shall present this dependence in our fnal computations only. The first equation implies an obvious inequality which is very useful below:

$$\lambda=\big((A+\varepsilon B)\varphi,\varphi\big). \tag{2.2}$$

By the general operator theory, we have the expansions

$$\lambda=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\dots,\ \varphi=\varphi_0+\varepsilon\varphi_1+\varepsilon^2\varphi_2+\dots, \tag{2.3}$$

where one should keep in mind in what follows that, in contrast to $\lambda_k$, the functions $\varphi_k$ are complex-valued in general. Our objective is to construct the simplest possible formulas for $\lambda_0$, $\lambda_1$, … , $\lambda_4$.

**Lemma 1.** *The following equations hold*:

$$\begin{gathered}\lambda_1=\big(\varphi_0,B\varphi_0\big), \quad \lambda_2=\big(\varphi_1,(B-\lambda_1)\varphi_0\big), \quad \lambda_3=\big(\varphi_1,(B-\lambda_1)\varphi_1\big),\\ \lambda_4=\big(\varphi_2,(B-\lambda_1)\varphi_1\big)+\lambda_2\big(\varphi_0,\varphi_2\big),\end{gathered} \tag{2.4}$$

*where the functions $\varphi_0$, $\varphi_1$, $\varphi_2$ can be found from the equations* (2.14)–(2.16) *and satisfy the conditions* (2.9)–(2.11).

In specific calculations, it is useful that all the $\lambda_k$ are real, according to the properties of eigenvalues of selfadjoint operators. Certainly, this fact can be verified directly.

**Proof.** Substitute the decomposition (2.3) into the equations and relations (2.1)–(2.2), perform the corresponding multiplications, and equate the coefficients at the powers of $\varepsilon$ to zero. By (2.2), this gives

$$\lambda_0=\big(A\varphi_0,\varphi_0\big),$$

$$\lambda_1=\big(B\varphi_0,\varphi_0\big)+\big(A\varphi_1,\varphi_0\big)+\big(A\varphi_0,\varphi_1\big) \tag{2.5}$$

$$\lambda_2=\big(B\varphi_1\,\varphi_0\big)+\big(A\varphi_2\,\varphi_0\big)+\big(B\varphi_0\,\varphi_1\big)+\big(A\varphi_1\,\varphi_1\big)+\big(A\varphi_0\,\varphi_2\big) \tag{2.6}$$

$$\begin{aligned}\lambda_3=&\big(B\varphi_2,\varphi_0\big)+\big(A\varphi_3,\varphi_0\big)+\big(B\varphi_1,\varphi_1\big)+\big(A\varphi_2,\varphi_1\big)\\ &+\big(B\varphi_0,\varphi_2\big)+\big(A\varphi_1,\varphi_2\big)+\big(A\varphi_0\varphi_3\big),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\lambda_4=&\big(B\varphi_3,\varphi_0\big)+\big(A\varphi_4,\varphi_0\big)+\big(B\varphi_2,\varphi_1\big)+\big(A\varphi_3,\varphi_1\big)\\ &+\big(B\varphi_1,\varphi_2\big)+\big(A\varphi_2,\varphi_2\big)+\big(B\varphi_0,\varphi_3\big)+\big(A\varphi_1,\varphi_3\big)+\big(A\varphi_0,\varphi_4\big),\end{aligned} \tag{2.8}$$

and, by the second relation in (2.2), we obtain

$$\big(\varphi_0,\varphi_0\big)=1, \tag{2.9}$$

$$\big(\varphi_1,\varphi_0\big)+\big(\varphi_0,\varphi_1\big)=0, \tag{2.10}$$

$$\big(\varphi_2,\varphi_0\big)+\big(\varphi_1,\varphi_1\big)+\big(\varphi_0,\varphi_2\big)=0, \tag{2.11}$$

$$\big(\varphi_3,\varphi_0\big)+\big(\varphi_2,\varphi_1\big)+\big(\varphi_1,\varphi_2\big)+\big(\varphi_0,\varphi_3\big)=0, \tag{2.12}$$

$$\big(\varphi_4,\varphi_0\big)+\big(\varphi_3,\varphi_1\big)+\big(\varphi_2,\varphi_2\big)+\big(\varphi_1,\varphi_3\big)+\big(\varphi_0,\varphi_4\big)=0, \tag{2.13}$$

while the first relation in (2.2) gives

$$A\varphi_0 = \lambda_0\varphi_0, \tag{2.14}$$
$$A\varphi_1 = \lambda_0\varphi_1 + \lambda_1\varphi_0 - B\varphi_0, \tag{2.15}$$
$$A\varphi_2 = \lambda_0\varphi_2 + \lambda_2\varphi_0 + \lambda_1\varphi_1 - B\varphi_1, \tag{2.16}$$
$$A\varphi_3 = \lambda_0\varphi_3 + \lambda_3\varphi_0 + \lambda_2\varphi_1 + \lambda_1\varphi_2 - B\varphi_2, \tag{2.17}$$

Let us "transfer" the operator $A$ in relations (2.5)–(2.21) to functions $\varphi_k$ with lesser indices and replace the expressions $A\varphi_k$ by the right-hand sides of (2.14)–(2.17). This gives

$$\lambda_1 = \big(B\varphi_0, \varphi_0\big) + \lambda_0\Big(\big(\varphi_1, \varphi_0\big) + \big(\varphi_0, \varphi_1\big)\Big) \overset{\text{by (2.10)}}{=\!=\!=} \big(B\varphi_0, \varphi_0\big), \tag{2.18}$$

$$\begin{aligned}\lambda_2 &= \big(B\varphi_1, \varphi_0\big) + \lambda_0\Big(\big(\varphi_2, \varphi_0\big) + \big(\varphi_0, \varphi_2\big)\Big) + \big(B\varphi_0, \varphi_1\big) + \big(\lambda_0\varphi_1 + \lambda_1\varphi_0 - B\varphi_0, \varphi_1\big) \qquad (2.19)\\ &= \big(B\varphi_1, \varphi_0\big) + \lambda_0\Big(\big(\varphi_2, \varphi_0\big) + \big(\varphi_0, \varphi_2\big) + \big(\varphi_1, \varphi_1\big)\Big) + \lambda_1\big(\varphi_0, \varphi_1\big) \overset{\text{by (2.11)}}{=\!=\!=} \big((B-\lambda_1)\varphi_1, \varphi_0\big),\end{aligned}$$

$$\begin{aligned}\lambda_3 &= \big(B\varphi_2, \varphi_0\big) + \lambda_0\Big(\big(\varphi_3, \varphi_0\big) + \big(\varphi_0, \varphi_3\big)\Big) + \big(B\varphi_1, \varphi_1\big) + \big(\varphi_2, \lambda_0\varphi_1 + \lambda_1\varphi_0 - B\varphi_0\big)\\ &\quad + \big(B\varphi_0, \varphi_2\big) + \big(\lambda_0\varphi_1 + \lambda_1\varphi_0 - B\varphi_0, \varphi_2\big) \qquad (2.20)\\ &= \lambda_0\Big(\big(\varphi_3, \varphi_0\big) + \big(\varphi_0, \varphi_3\big) + \big(\varphi_1, \varphi_2\big) + \big(\varphi_2, \varphi_1\big)\Big) + \big(B\varphi_1, \varphi_1\big) + \lambda_1\Big(\big(\varphi_2, \varphi_0\big) + \big(\varphi_0, \varphi_2\big)\Big)\\ &\overset{\text{by (2.10) and (2.11)}}{=\!=\!=\!=\!=} \big((B-\lambda_1)\varphi_1, \varphi_1\big),\end{aligned}$$

$$\begin{aligned}\lambda_4 &= \big(B\varphi_3, \varphi_0\big) + \lambda_0\Big(\big(\varphi_4, \varphi_0\big) + \big(\varphi_0, \varphi_4\big)\Big) + \big(B\varphi_2, \varphi_1\big) + \big(\varphi_3, \lambda_0\varphi_1 + \lambda_1\varphi_0 - B\varphi_0\big)\\ &+ \big(B\varphi_1, \varphi_2\big) + \big(\lambda_0\varphi_2 + \lambda_2\varphi_0 + \lambda_1\varphi_1 - B\varphi_1, \varphi_2\big) + \big(B\varphi_0, \varphi_3\big) + \big(\lambda_0\varphi_1 + \lambda_1\varphi_0 - B\varphi_0, \varphi_3\big)\\ &= \lambda_0\Big(\big(\varphi_4, \varphi_0\big) + \big(\varphi_3, \varphi_1\big) + \big(\varphi_2, \varphi_2\big) + \big(\varphi_1, \varphi_3\big) + \big(\varphi_0, \varphi_4\big)\Big) + \big(B\varphi_2, \varphi_1\big) + \lambda_2\big(\varphi_0, \varphi_2\big)\\ &+ \lambda_1\Big(\big(\varphi_3, \varphi_0\big) + \big(\varphi_1, \varphi_2\big) + \big(\varphi_0, \varphi_3\big)\Big) \overset{\text{by (2.12) and (2.13)}}{=\!=\!=\!=\!=} \big((B-\lambda_1)\varphi_2, \varphi_1\big) + \lambda_2\big(\varphi_0, \varphi_2\big).\end{aligned} \tag{2.21}$$

Transferring now the (self-adjoint) operator $B - \lambda$ to the second factor in the formulas for $\lambda_k$, we obtain the assertion of the lemma.

### *2.1. Expansion of the Function $L$ with Respect to the Variable $p$ Up to $O(|p|^5)$*

Let us now apply the above formulas to our problem. Consider the equation for the torus $\mathbb{T}$ and the following "problem for a cell,"

$$\triangle_y^\Theta f = F, \quad \langle F\rangle_{\mathbb{T}} = 0, \tag{2.22}$$

where $F(y, x, p)$ is a smooth function $2\pi$-periodic with respect to the variables $y_j$ and has zero mean. We have already noted above that the minimal nondegenerate eigenvalue of the operator $\triangle_y^\Theta$ is equal to zero, and one can and may conveniently choose the corresponding normalized eigenfunction $\chi_0$ to be identically equal to one[2]. The condition that $F$ and $\chi_0(x, p, y) = 1$ are orthogonal means that the mean value $\langle F\rangle_{\mathbb{T}}$ vanishes. For the sake of completeness of our presentation, we state and prove the following lemma, which is a special case of Lemma 2 in [1].

**Lemma 2.** *For any $x \in K$ and any sufficiently small $p$, there exists a unique solution $f \in C^\infty(\mathbb{T})$ of the equation*

$$(\mathcal{H} - L)f = F \tag{2.23}$$

*with zero mean $\langle f\rangle_{\mathbb{T}}$ (the condition that $f$ is orthogonal to the function $\chi_0(x, p, y)$).*

[2]One can also set $\chi_0 = e^{iq(x,p)}$, $q(x, p) \in \mathbb{R}$; however, this is inconvenient for many reasons.

*If $F$ is a smooth function of the variables $x \in K$, $y \in \mathbb{T}$, and $p$ (for small $p$) and some additional parameters $z$, then the solution $f(x,p,y,z)$ is also a smooth function. Any other smooth solution $f_1(x,p,y,z)$ of equation* (2.23) *on the torus $\mathbb{T}$ can be expressed using $f$ from the formula $f_1 = f+g$, where $g(x,p,z)$ is a smooth function of the parameters $(x,p,z)$.*

**Proof.** It follows from the general theory of elliptic equations on compact manifolds that $\mathcal{H}_0 - L$ defines a Fredholm operator from the Sobolev space $H^2(\mathbb{T})$ to $L^2(\mathbb{T})$. Therefore, the solvability condition for equation (2.23) is that the right-hand side must be orthogonal to the solution $\chi_0$ of the homogeneous equation, and the solution $f$ orthogonal to $\chi_0$ is uniquely defined.

One can readily prove, using the general theory of elliptic operators on compact manifolds (see, e.g., [7]) that the functions entering these solutions are infinitely differentiable. To this end, consider the following problem: for chosen $(x,p)$, a given function $F(y)$, and a number $d$, on must find a function $u(y)$ and a number $g$ satisfying the equations

$$(\mathcal{H} - L)u(y) - g\chi_0(x,p,y) = F(y), \qquad (u(y),\chi_0)_{L^2(\mathbb{T})} = d. \tag{2.24}$$

Since the operator $A(x,p)$ corresponding to this problem, i.e., $A(x,p)\colon \mathcal{H}^{s+2}(\mathbb{T})\times C^1 \to \mathcal{H}^s(\mathbb{T})\times C^1$, where $C^1$ stands for the one-dimensional complex space, is invertible and $A(x,p)$ depends smoothly on the parameters $(x,p)$, it follows that the inverse operator $A^{-1}(x,p)$ also depends smoothly on the parameters $(x,p)$. Taking the function $F(x,p,y,z)$ for $F$, we see that the solution $u(x,p,y,d,z)$ of problem (2.24) depends smoothly on the parameters $(x,p,z)$. However, $f(x,p,y,z)$ is just the solution of problem (2.24) for $d=0$ and $g=0$, and hence $f(x,p,y,z)$ is a smooth function of the parameters $(x,p,z)$ with values in the space $\mathcal{H}^{s+2}(\mathbb{T})$ for any $s$. Using standard embedding theorems for Sobolev spaces, we now see that $f(x,p,y,z)$ is a jointly infinitely differentiable function of its variables.

Denote the solution defined in the lemma by

$$f(y,x,p) = \frac{1}{\triangle^{\Theta}_y} F(y,x,p) \tag{2.25}$$

In addition to the operator $D = \langle p, \Theta(x)\nabla_y\rangle$ linearly depending on $p$, we also introduce the operator

$$Q = DC^2 - C^2 D, \tag{2.26}$$

which also linearly depends on $p$. Denote by $g_0(y,x)$, $g_1(y,x,p)$, $g_2(y,x,p)$ solutions with zero means for the problem on a cell,

$$g_0 = \frac{1}{\triangle^{\Theta}_y}\tilde{a}, \quad g_1 = \frac{1}{\triangle^{\Theta}_y}(D\tilde{a}), \quad g_2 = \frac{1}{\triangle^{\Theta}_y}(Qg_1 - \langle Qg_1\rangle_{\mathbb{T}}), \quad \langle g_{1,2,3}\rangle_{\mathbb{T}} = 0. \tag{2.27}$$

Note that $g_1(y,x,p)$ is a linear homogeneous function of $p$ and $g_2(y,x,p)$ is a second-order homogeneous polynomial in $p$.

**Proposition 1** (the formulas below were announced in [1])**.** *For $x$ belonging to a compact set $K$ and for sufficiently small $p$, the minimal eigenvalue $L(x,p)$ of the operator $\mathcal{H}$ is nondegenerate and analytic with respect to $p$, and the normalized function $\chi(x,p,y)$ can also be chosen to be analytic in $p$, in such a way that the equation $L(x,p) = L^{(2)}(x,p) + L^{(4)}(x,p) + O(|p|^6)$ holds, where*

$$L^{(2)}(x,p) = p^2{C_0}^2 - \langle Qg_1\rangle_{\mathbb{T}} = p^2{C_0}^2 + \langle g_1 D\tilde{a}\rangle_{\mathbb{T}}, \tag{2.28}$$

$$L^{(4)}(x,p) = p^4\langle g_0\tilde{a}\rangle_{\mathbb{T}} + 2p^2\langle g_1 Q g_0\rangle_{\mathbb{T}} + \langle {g_1}^2\rangle_{\mathbb{T}}\langle Qg_1\rangle_{\mathbb{T}} + p^2\langle {g_1}^2\tilde{a}\rangle_{\mathbb{T}} + \langle g_2 Q g_1\rangle_{\mathbb{T}}, \tag{2.29}$$

*and the following relations hold for $\chi(x,p,y)$:*

$$\chi = 1 - ig_1(y,x,p) + p^2 g_0(y,x) - g_2(y,x,p) - \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}} + O(|p|^3), \tag{2.30}$$

$$\|1 - ig_1(y,x,p) + p^2 g_0(y,x) - g_2(y,x,p) - \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}}\| = 1 + O(|p|^3). \tag{2.31}$$

**Proof.** Let us apply formulas (2.4) to our problem. Write

$$A = -\triangle_y^\Theta, \quad \varepsilon B = -i(DC^2(y,x) + C^2(y,x)D) + C^2(y,x)p^2,$$

Here $\varphi = \chi$ and $\lambda = L$, and we obtain in fact an expansion in powers of $|p|$, which we represent in the form

$$\chi = \chi_0 + \chi_1 + \chi_2 + \dots, \qquad L = L_0 + Ł_1 + L_2 + \dots,$$

assuming that $\chi_k$ and $L_k$ are of the order of $O(|p|^k)$. For the inner product $(\cdot,\cdot)$ we take the inner product on the torus $\mathbb{T}$ and use the notation (4.21) for the mean value on the torus $\mathbb{T}$.

For $\varphi_0 = \chi_0$ and $\lambda_0 = L_0$ we choose

$$\varphi_0 = 1, \quad \lambda_0 = L_0 = 0.$$

This immediately gives

$$\varepsilon\lambda_1 = p^2\langle C^2\rangle_T \equiv p^2 C_0^2, \tag{2.32}$$

i.e., $\varepsilon\lambda_1$ is found with an improved accuracy.

Further, by the definition of the operator $D$ and by the equation for the function $\chi_1$, we have

$$\begin{aligned}\varepsilon(B-\lambda_1)\varphi_0 &= -iD(C^2(y,x)) + p^2\tilde{a}(y,x) = (-iD+p^2)\tilde{a}(y,x),\\ \varepsilon\varphi_1 &= \frac{1}{\triangle_y^\Theta}\Big((-iD+p^2)\tilde{a}\Big) = -ig_1(y,x,p) + p^2 g_0(y,x).\end{aligned}$$

For subsequent calculations, it is useful to recall that the numbers $\varepsilon\lambda_k$, $g$, and $Q$ are real. The second formula in (2.4) gives

$$\varepsilon^2\lambda_2 = \varepsilon^2(\varphi_1,(B-\lambda_1)\varphi_0) = p^4(g_0,\tilde{a}) + (g_1, D\tilde{a}) = \langle g_1 D\tilde{a}\rangle_{\mathbb{T}} + p^4\langle g_0\tilde{a}\rangle_{\mathbb{T}}.$$

Let us now evaluate the quantity $\varepsilon^3\lambda_3 = \varepsilon^3\big(\varphi_1,(B-\lambda_1)\varphi_1\big)$ up to accuracy $O(|p|^6)$. Substitute the expressions obtained above into this formula and take into account that $\varepsilon\lambda_3$, $g_0$, and $Q$ are real and that the adjoint operator is $Q^* = -Q$. This gives

$$\begin{aligned}\varepsilon^3\lambda_3 &= p^2(g_1,Qg_0) - p^2(g_0,Qg_1) + p^2(g_1,\tilde{C}^2 g_1) + p^6(g_0,\tilde{a}(y,x)g_0)\\ &= 2p^2(g_1,Qg_0) + p^2(g_1,\tilde{a}g_1) + O(|p|^6) = 2p^2\langle g_1,Qg_0\rangle_{\mathbb{T}} + p^2\langle g_1^2\tilde{a}\rangle_{\mathbb{T}}.\end{aligned} \tag{2.33}$$

Finally let us evaluate the quantity $\varepsilon^4\lambda_4 = \varepsilon^4\big(\varphi_2,(B-\lambda_1)\varphi_1\big) + \varepsilon^4\lambda_2\big(\varphi_0,\varphi_2\big)$. We have $\varepsilon^2\varphi_2 = -\frac{1}{\triangle_y^\Theta}(\varepsilon^2\lambda_2 - \varepsilon^2(B-\lambda_1)\varphi_1) + R$. Here $R$ stands for the constant (with respect to the variables $y$) which is defined by condition (2.11). With regard to the definition of the operator $\frac{1}{\triangle_y^\Theta}$, we obtain

$$R = -\frac{\varepsilon^2}{2}(\varphi_1,\varphi_1) = -\frac{1}{2}((g_1,g_1) + p^4(g_0,g_0)),$$

$$\begin{aligned}\varepsilon^2\varphi_2 &= -\frac{1}{\triangle_y^\Theta}\Big(\langle g_1 D\tilde{C}^2\rangle_{\mathbb{T}} + p^4\langle g_0\tilde{a}\rangle_{\mathbb{T}} + Qg_1 - p^4\tilde{a}g_0 + ip^2 Qg_0 + ip^2 g_1\tilde{a}\Big)\\ -\frac{1}{2}((g_1,g_1) + p^4(g_0,g_0)) &= -\frac{1}{\triangle_y^\Theta}\Big(Qg_1 + \langle g_1 D\tilde{a}\rangle_{\mathbb{T}}\Big) - \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}} + O(|p|^3),\\ \varepsilon^2(\varphi_0,\varphi_2) &= R = -\frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}} + O(|p|^4).\end{aligned} \tag{2.34}$$

We can similarly find $\varepsilon^2(B-\lambda_1)\varphi_1 = -Qg_1 + O(|p|^3)$ and

$$\varepsilon^4\lambda_4 = \Big(\frac{1}{\triangle_y^\Theta}(Qg_1 + \langle g_1 D\tilde{a}\rangle_{\mathbb{T}}) + \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}}, Qg_1\Big) - \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}}\langle g_1 D\tilde{a}\rangle_{\mathbb{T}} + O(|p|^6).$$

Note that $\langle Qg_1\rangle_{\mathbb{T}} = -\langle g_1 D\tilde{a}\rangle_{\mathbb{T}}$, and therefore we have

$$\begin{aligned}\varepsilon^4\lambda_4 &= \Big(\frac{1}{\triangle_y^\Theta}(Qg_1 + \langle g_1 D\tilde{C}^2\rangle_{\mathbb{T}}), Qg_1\Big) - \langle g_1^2\rangle_{\mathbb{T}}\langle g_1 D\tilde{a}\rangle_{\mathbb{T}} + O(|p|^6)\\ &= \langle g_2 Qg_1\rangle_{\mathbb{T}} + \langle g_1^2\rangle_{\mathbb{T}}\langle Qg_1\rangle_{\mathbb{T}} + O(|p|^6).\end{aligned} \tag{2.35}$$

Adding now $\varepsilon^k\lambda_k$, $k = 0,\dots,4$, and $\varepsilon^k\varphi_k$, $k = 0,\dots,4$, we obtain (2.28) and (2.29).

One can prove that, in fact, the expansion of $L_0$ is in homogeneous polynomials in $p_k$ of even degrees.

## 3. FORMULAS FOR $L^{(2)}$ AND $L^{(4)}$ IN THE SINGLE-PHASE CASE

Let us begin with the case $n = 1$. Then the formulas of Proposition 1 can be realized as quadratures. We have

$$\triangle_y^\Theta = \Theta_x^2\frac{\partial}{\partial y}C^2(y,x)\frac{\partial}{\partial y}, \quad D = \langle p, \Theta_x\rangle\frac{\partial}{\partial y}, \quad Q = \langle p, \Theta_x\rangle(\frac{\partial}{\partial y}C^2 + C^2\frac{\partial}{\partial y}).$$

Let $f(y,x)$ be a function $2\pi$-periodic with respect to $y$ with zero mean. Denote by $\hat{I}$ the operator of integration with respect to the variable $y$, which gives functions with zero mean, namely.

$$\hat{I}f(y,x) = \int_0^y f(y,x)\,dy - \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^y f(\eta,x)\,d\eta\Big)\,d\eta.$$

It is clear that $\frac{\partial}{\partial y}\hat{I}f = \hat{I}\frac{\partial}{\partial y}f$ for any function $f$ with zero mean. We have further

$$(\triangle_y^\Theta)^{-1}f = \frac{1}{\Theta_x^2}\hat{I}\left(\frac{\hat{I}f + \beta_f(x)}{C^2(y,x)}\right), \quad \beta_f(x) = -\int_0^{2\pi}\frac{\hat{I}f\,dy}{C^2(y,x)}\Bigg/\int_0^{2\pi}\frac{dy}{C^2(y,x)}. \tag{3.1}$$

Indeed, $F = (\triangle_y^\Theta)^{-1}f$ is a solution of the equation $\triangle_y^\Theta F = f$ with zero mean, and hence the relations

$$C^2\frac{\partial}{\partial y}F = \frac{1}{\Theta_x^2}(\hat{I}f + \beta(x)), \qquad \frac{\partial}{\partial y}F = \frac{1}{\Theta_x^2}\frac{\hat{I}f + \beta(x)}{C^2(y,x)} \tag{3.2}$$

hold for some function $\beta(x)$. Taking the mean of both the sides of the last equation, we obtain

$$\Big\langle\frac{\hat{I}f}{C^2}\Big\rangle_{\mathbb{T}} + \beta(x)\Big\langle\frac{1}{C^2}\Big\rangle_{\mathbb{T}} = 0, \qquad \beta(x) = -\Big\langle\frac{\hat{I}f}{C^2}\Big\rangle_{\mathbb{T}}\Big/\Big\langle\frac{1}{C^2}\Big\rangle_{\mathbb{T}} = 0. \tag{3.3}$$

Applying the operator $\hat{I}$ to the latter relation in (3.2), we obtain (3.1) with regard to the second relation in (3.3).

Let us proceed with the calculation of the functions $g_k$ in (2.27). Applying formula (3.1) to the function $f = \tilde{a}$, we see that

$$g_0 = \frac{1}{\Theta_x^2}\hat{I}\left(\frac{\hat{I}\tilde{a} + \beta_0(x)}{C^2(y,x)}\right), \quad \beta_0(x) = -\int_0^{2\pi}\frac{\hat{I}\tilde{a}\,dy}{C^2(y,x)}\Bigg/\int_0^{2\pi}\frac{dy}{C^2(y,x)}. \tag{3.4}$$

To obtain an expression for $g_1$, we apply formula (3.1) to the function $f = \langle p, \nabla_y^\Theta\rangle \frac{\partial}{\partial y}\tilde{a}$. Since $\hat{I}\frac{\partial}{\partial y}\tilde{a} = \tilde{a}$, it follows that

$$g_1 = \frac{\langle p, \Theta_x\rangle}{\Theta_x^2}\hat{I}\Big(1 + \frac{\gamma_1}{C^2}\Big), \quad \gamma_1 = -2\pi \Big/ \int_0^{2\pi} \frac{dy}{C^2(y,x)}\ . \tag{3.5}$$

To obtain an expression for $g_2$, we apply formula (3.1) to the function $f = Qg_1 - \langle Qg_1\rangle_{\mathbb{T}}$. Since

$$\begin{aligned} Qg_1 - \langle Qg_1\rangle_{\mathbb{T}} &= \langle p, \Theta_x\rangle(\frac{\partial}{\partial y}C^2 + C^2\frac{\partial}{\partial y})g_1 - \langle p, \Theta_x\rangle\langle C^2\frac{\partial}{\partial y}g_1\rangle_{\mathbb{T}} = \langle p, \Theta_x\rangle\frac{\partial}{\partial y}(C^2 g_1) \\ &+ \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}(C^2 + \gamma_1) - \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}(\langle C^2\rangle_{\mathbb{T}} + \gamma_1) = \langle p, \Theta_x\rangle\frac{\partial}{\partial y}(C^2 g_1) + \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}\hat{C}^2, \end{aligned} \tag{3.6}$$

it follows that

$$\begin{aligned} g_2 = (\triangle_y^\Theta)^{-1}(Qg_1 - \langle Qg_1\rangle_{\mathbb{T}}) &= \frac{\langle p, \Theta_x\rangle}{\Theta_x^2}\hat{I}g_1 + \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}g_0 \\ &= \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^4}\Big[\hat{I}\Big(\hat{I}\Big(1 + \frac{\gamma_1}{C^2}\Big)\Big) + \hat{I}\Big(\frac{\hat{I}\tilde{a} + \beta_0(x)}{C^2(y,x)}\Big)\Big], \end{aligned} \tag{3.7}$$

where $\gamma_1$ is expressed by the second formula in (3.5). Now to express $L^{(2)}(x,p)$ and $L^{(4)}(x,p)$ in quadratures, it suffices to substitute the expressions for $g_k$ obtained in (3.4), (3.5), and (3.7) into formulas (2.28) and (2.29). Here formula (2.28) becomes

$$\begin{aligned} L^{(2)}(x,p) &= p^2\langle C^2\rangle_{\mathbb{T}} - \langle Qg_1\rangle_{\mathbb{T}} = p^2\langle C^2\rangle_{\mathbb{T}} - \langle p, \Theta_x\rangle\langle C^2\frac{\partial}{\partial y}g_1\rangle_{\mathbb{T}} \\ &= p^2\langle C^2\rangle_{\mathbb{T}} - \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}(\langle C^2\rangle_{\mathbb{T}} + \gamma_1) = p^2\langle C^2\rangle_{\mathbb{T}} + \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}((\langle C^{-2}\rangle_{\mathbb{T}})^{-1} - \langle C^2\rangle_{\mathbb{T}}). \end{aligned} \tag{3.8}$$

We do not present the corresponding formula for $L^{(4)}(x,p)$ here, because it is too cumbersome.

Consider an example for which $C^2 = C_0^2(x) + \alpha(x)\cos y$ and $|\alpha(x)| < C_0^2(x)$. In this case, the means in the last equation can be evaluated explicitly, and we obtain

$$L^{(2)}(x,p) = |p|^2 C_0^2(x) - \frac{\langle p, \Theta_x\rangle^2}{\Theta_x^2}(C_0^2(x) - \sqrt{C_0^4(x) - \alpha^2(x)}). \tag{3.9}$$

## 4. PERTURBATION THEORY WITH RESPECT TO THE SMALL OSCILLATING PART OF THE FUNCTION $C^2$

Let $\delta > 0$ be some number (which is assumed below to be sufficiently small). Suppose that

$$\tilde{a} = \delta a(y,x)$$

in (1.6), i.e., $C^2(y,x) = C_0^2(x) + \delta a(y,x)$, where $a(y,x)$ is a smooth real function $2\pi$-periodic with respect to the variables $y_j$ and has zero mean, and $C_0^2(x) > 0$ for any $x$ belonging to the compact set $\mathcal{K}$. Then $C^2(y,x) > 0$ for $|\delta| \leqslant \rho$ for some $\rho > 0$, and it follows from Lemma 3 and from Lemmas 2 and 3 in [1] that the problem on a cell (2.22) has a smooth solution $f(y,x,p)$ with zero mean on the torus $\mathbb{T}$ for any chosen $x$ and $\delta$ and for any smooth function $F(y,x,p)$ on the torus $\mathbb{T}$ with zero mean, and the operator $(\triangle_y^\Theta)^{-1}$ is well defined with acts from the space of functions with zero mean values on the torus $\mathbb{T}$ and is inverse in this space to the operator $\triangle_y^\Theta$, and thus $f = (\triangle_y^\Theta)^{-1}F$. Since the operator $\triangle_y^\Theta$ now depends not only on the parameters $x$ but also on the parameter $\delta$, it follows that, for the case in which $F$ depends on $(y,x,p,\delta)$, the function $f$ depends also on $(y,x,p,\delta)$. Assume that the function $F(y,x,p,\delta)$ smoothly depends $(y,x,p,\delta)$ and is analytic in $\delta$ for $y \in \mathbb{T}$, $x \in K$, and $|\delta| \leqslant \rho$. Here we keep in mind that the function $F(y,x,p,\delta)$ can be analytically continued to some neighborhood $V$ of the interval $[-\rho,\rho]$ of the real axis on the complex plane of the variable $\delta$ and smoothly depends on $(y,x,p,\delta)$ for $y \in \mathbb{T}$, $x \in \mathcal{K}$, $|p| < p_0$, and $\delta \in V$.

**Lemma 3.** *If $F(y,x,p,\delta)$ is a smooth function of $(y,x,p,\delta)$ for $y\in\mathbb{T}$, $x\in K$, $|p|<p_0$, and $|\delta|\leqslant\rho$ and analytic in $\delta$, then the solution $f(y,x,p,\delta)$ of the problem on a cell* (2.22) *is a smooth function of $(y,x,p,\delta)$ for $y\in\mathbb{T}$, $x\in K$, $|p|<p_0$, and $|\delta|\leqslant\rho$ and analytic in $\delta$.*

**The proof** of this assertion is now carried out similarly to the proof of Lemma 3 here and of Lemma 2 in [1]. To indicate the dependence of the operator $\triangle_y^\Theta$ on $x$ and $\delta$, we write $A(x,\delta)=\triangle_y^\Theta$. With regard to the parameter $\delta$, we have

$$A(x,\delta)=A_0+\delta B, A_0\,(x,\partial/\partial y)=C_0^2(x)\langle\nabla_y^\Theta,\nabla_y^\Theta\rangle,\qquad B(y,x,\partial/\partial y)=\langle\nabla_y^\Theta,a(y,x)\nabla_y^\Theta\rangle,\quad (4.1)$$

1. Let us show first that the function $f(y,x,p,\delta)$ can be regarded as a smooth function of the parameters $(x,p,\delta)$ for $x\in K$, $|p|<p_0$, and $|\delta|\leqslant\rho$ with values in the Sobolev space $\mathcal{H}^2(\mathbb{T})$ of function which belong to $L_2(\mathbb{T})$ together with the generalized derivatives distribution of the first and second order. Indeed, if $A$ is regarded for chosen $x$ and $\delta$ as a continuous operator from the space $\widetilde{\mathcal{H}}^2(\mathbb{T})$ consisting of functions in $\mathcal{H}^2(\mathbb{T})$ to the space $\widetilde{L}_2(\mathbb{T})$, which is the subspace of $L_2(\mathbb{T})$ consisting of the functions with zero means, then $A$ is invertible, because it follows from the general theory of elliptic operators on compact manifolds (see, e.g., [7]) that $A$ is a Fredholm operator from $\mathcal{H}^2(\mathbb{T})$ to $L_2(\mathbb{T})$, and the condition that the mean value of a function vanishes is the very solvability condition for equation (2.22). Since the operator $A$ smoothly depends on the parameters $(x,\delta)$, it follows that the inverse operator $A^{-1}$ has the same property, which can readily be established by using the well-known formula for the inverse operator,

$$A^{-1}(x,\delta)=A^{-1}(x_0,\delta_0)\sum_{j=0}^{\infty}((A(x_0,\delta_0)-A^{-1}(x,\delta))A^{-1}(x_0,\delta_0))^j,\tag{4.2}$$

where $x_0\in K$ and $|\delta_0|\leqslant\rho$, where this series converges uniformly with respect to the operator norm for all $(x,\delta)$ in a sufficiently small neighborhood of the point $(x_0,\delta_0)$ and admits termwise differentiation with respect to the parameters $(x,\delta)$. This implies that $f(y,x,p,\delta)=A^{-1}F$ is a smooth function of the parameters $(x,p,\delta)$ for $x\in K$, $|p|<p_0$, $|\delta|\leqslant\rho$ with the values in $\widetilde{\mathcal{H}}^2(\mathbb{T})$.

2. Let $x_0\in K$, and let $\delta_0$ be real with $|\delta_0|\leqslant\rho$. If $x,p$ are in a sufficiently small neighborhood of the point $x_0\in K$, then the series (4.2) converges also for complex values $\delta$ with $|\delta-\delta_0|<\rho_1$ for some $\rho_1>0$, and hence there is a neighborhood of the segment $[-\rho,\rho]$ of the real axis such that the operator $A\colon\widetilde{\mathcal{H}}^2(\mathbb{T})\to\widetilde{L}_2(\mathbb{T})$ is invertible. By the standard smoothness theorems for solutions of elliptic equations, this implies the invertibility of $A$ for these $(x,\delta)$ if $A$ is regarded as an operator $A\colon\widetilde{\mathcal{H}}^{s+2}(\mathbb{T})\to\widetilde{\mathcal{H}}^s(\mathbb{T})$ for any $s\geqslant 0$ (the tilde means here that the corresponding subspaces of functions with zero mean values are considered). In this case, for any $s\geqslant0$ and for $x_0\in K$ and $\delta_0\in U$, the series converges uniformly with respect to the operator norm in some sufficiently small neighborhood of the point $(x_0,\delta_0)$ and admits the termwise differentiation with respect to the parameters $(x,p,\delta)$, which implies that $f(y,x,p,\delta)=A^{-1}F$ is a smooth function of the parameters $(x,p,\delta)$ for $x\in K$, $|p|<p_0$, $\delta\in U\cap V$ which is analytic in $\delta$ and takes the values in $\widetilde{\mathcal{H}}^{s+2}(\mathbb{T})$.

3. It follows now from the standard embedding theorems for Sobolev spaces that $f(y,x,p,\delta)$ is a smooth function of $(y,x,p,\delta)$ for $y\in\mathbb{T}$, $x\in K$, $\delta\in U\cap V$ which is analytic in $\delta$. This completes the proof of the lemma.

The lemma implies that $g_0$, $g_1$, and $g_2$ (introduced in (2.27)) are smooth functions of $(y,x,p,\delta)$ for $y\in\mathbb{T}$, $x\in K$, $|\delta|\leqslant\rho$ which are analytic in $\delta$. To find approximations for these functions, we can apply perturbation theory, namely, expand $g_k$ in series in powers of the parameter $\delta$,

$$g_k=\delta g_{k,1}(y,x,p)+\delta^2 g_{k,2}(y,x,p)+\dots\tag{4.3}$$

and seek the coefficients $g_{k,j}$ of these expansions using the corresponding recurrence relations. Recall that both $g_k$ and the coefficients $g_{k,j}$ are $2\pi$-periodic functions, with respect to the variables $y_j$, with zero mean value which are homogeneous polynomials of degree $k$ with respect to the variables $p$. Note that the expansions (4.3) admit termwise differentiation with respect to the variables $(y, x, p)$. Below we do no use the fact that $g_k$ is analytic in $\delta$, and (4.3) can be regarded as asymptotic expansions.

Representing the equation for $g_0$ in the form $A_0(x, \partial/\partial y)g_0 + \delta B(y, x, \partial/\partial y)g_0 = \delta a(y, x)$, we see that $g_{0,j}$ can be found from the recurrence relations $A_0 g_{0,1} = a$, $A_0 g_{0,j+1} + B g_{0,j} = 0$ for $j \geqslant 1$. Solving these equations, we formally obtain

$$g_{0,1} = A_0^{-1}a, \quad g_{0,2} = -A_0^{-1}BA_0^{-1}a, \quad g_{0,3} = A_0^{-1}BA_0^{-1}BA_0^{-1}a, \dots, \tag{4.4}$$

where (we would like to recall) the symbol $A_0^{-1}F$ with $F$ having zero mean value stands for the solution of the equation $A_0 f = F$ with zero mean value of $f$ (all functions are $2\pi$-periodic with respect to $y_j$). Thus, $g_0 = \delta g_{0,1} + O(\delta^2) = \delta A_0^{-1}a + O(\delta^2)$. Similarly, using the equation $A_0(x, \partial/\partial y)g_1 + \delta B(y, x, \partial/\partial y)g_1 = \delta Da(y, x)$, we see that

$$g_{1,1} = A_0^{-1}Da, \quad g_{1,2} = -A_0^{-1}BA_0^{-1}Da, \quad g_{1,2} = A_0^{-1}BA_0^{-1}BA_0^{-1}Da, \dots, \tag{4.5}$$

and $g_1 = \delta DA_0^{-1}a + O(\delta^2)$. Here we use the fact that the operators $A_0$ and $D$ commute, because they are differential operators with respect to the variables $y$ whose coefficients are constant with respect to $y$, and hence, the operators $A_0^{-1}$ and $D$ also commute. Since $g_1$ is a second-order homogeneous polynomial in $p$, it follows that here $O(\delta^2)$ denotes a linear homogeneous function of $p$ whose coefficients are of the form $O(\delta^2)$. The equation

$$A_0(x, \partial/\partial y)g_2 + \delta B(y, x, \partial/\partial y)g_2 = Qg_1 - \langle Qg_1\rangle_{\mathbb{T}} = 2\delta C_0^2 Dg_{1,1} + O(\delta^2) = 2\delta C_0^2 D^2 A_0^{-1}b + O(\delta^2),$$

where $O(\delta^2)$ stands for a second-order homogeneous polynomial in $p$ with coefficients of the form $O(\delta^2)$, now implies that $g_{2,1} = 2C_0^2(x)D^2(A_0^{-1})^2 a$, and thus $g_2 = 2\delta C_0^2(x)D^2(A_0^{-1})^2 a + O(\delta^2)$.

We also note the relations

$$Qg_0 = 2\delta C_0^2(x)Dg_{0,1} + O(\delta^2) = 2\delta C_0^2(x)DA_0^{-1}a + O(\delta^2), \tag{4.6}$$

$$Qg_1 = 2\delta C_0^2 Dg_{1,1} + \delta^2(Da + aD)g_{1,1} + 2\delta^2 C_0^2 Dg_{1,2} + O(\delta^3), \tag{4.7}$$

$$\langle Qg_1\rangle_{\mathbb{T}} = \delta^2\langle aDg_{1,1}\rangle_{\mathbb{T}} + O(\delta^3) = \delta^2\langle aD^2A_0^{-1}a\rangle_{\mathbb{T}} + O(\delta^3), \tag{4.8}$$

which are used below.

The solution with zero mean, $f = A_0^{-1}F$, of the equation $A_0 f = F$ for a function $F(y, x)$ with zero mean can be obtained by expanding $F$ in the Fourier series

$$F = \sum_{\nu\neq 0} F_\nu(x)\exp\big(i\langle \nu, y\rangle\big), \tag{4.9}$$

where $\nu$ stands for the integer multi-index $\nu = (\nu_1, \dots, \nu_n)$ with integer components $\nu_j \geqslant 0$. The solution is of the form

$$f = A_0^{-1}f = -C_0^{-2}(x)\sum_{\nu\neq 0}\langle \Theta_x\nu, \Theta_x\nu\rangle^{-1}F_\nu \exp\big(i\langle \nu, y\rangle\big), \tag{4.10}$$

If $F$ is a finite sum of the form (4.9), then $f$ is also a finite sum of the form (4.10). The action of the operator $B$ is the multiplication by a finite sum. This implies that, if $a$ is a finite sum of the form (4.9), then all coefficients $g_{k,l}$ can be found in the form of finite sums of this kind. Therefore, the evaluation of the coefficients $g_{k,l}$ reduces in these cases to purely algebraic operations with the Fourier coefficients for $a$ and the entries of the matrix $\Theta_x$.

We can now obtain approximate values for $L^{(2)}$ and $L^{(4)}$ in formulas (2.28) and (2.29). Note also that the operator $A_0^{-1}$ is symmetric, i.e., the relation $\langle f_1 A_0^{-1} f_2\rangle_{\mathbb{T}} = \langle f_2 A_0^{-1} f_1\rangle_{\mathbb{T}}$ holds for any functions $f_1$ and $f_2$ that are $2\pi$-periodic with respect to $y_j$ and have zero mean values. Indeed,

$\langle f_1 A_0^{-1} f_2\rangle_{\mathbb{T}} = \langle (A_0 A_0^{-1} f_1) A_0^{-1} f_2\rangle_{\mathbb{T}} = \langle (A_0^{-1} f_1) A_0 A_0^{-1} f_2\rangle_{\mathbb{T}} = \langle (A_0^{-1} f_1) f_2\rangle_{\mathbb{T}}$, and hence both the operator $A_0^{-1}$ and the operator $D^2$ can be transferred in the summands of formula (4.13) from one factor to another (the operator $D$ can be transferred together with the alternation of the sign). Using these properties and replacing the second summand in $L^{(2)}$ by the right-hand side of (4.8), we obtain

$$L^{(2)}(x,p) = p^2 C_0^2(x) - \delta^2 \langle aD^2 A_0^{-1} a\rangle_{\mathbb{T}} \tag{4.11}$$

$$-\delta^3 \langle (A_0^{-1} Da)(BA_0^{-1} Da)\rangle_{\mathbb{T}} + \delta^4 \langle (BA_0^{-1} Da)(A_0^{-1} BA_0^{-1} Da)\rangle_{\mathbb{T}} + O(\delta^5), \tag{4.12}$$

where $O(\delta^5)$ is a second-order homogeneous polynomial in $p$ with the coefficients of the form $O(\delta^5)$. In the same way, replacing $g_0$, $Qg_0$, $g_1$, $Qg_1$, and $g_2$ in formula (2.29) for $L_0^{(4)}$ by $\delta g_{0,1}$, $2\delta C_0^2 D g_{0,1}$, $\delta g_{1,1}$, $2\delta C_0^2 D g_{1,1}$, and $\delta g_{2,1}$, respectively, using (4.6) and (4.7), we obtain

$$L^{(4)} = \delta^2 (p^4 \langle g_{0,1} b\rangle_{\mathbb{T}} + 4p^2 \langle g_{1,1} C_0^2 D g_{0,1}\rangle_{\mathbb{T}} + 2\langle g_{2,1} C_0^2 D g_{1,1}\rangle_{\mathbb{T}}) + O(\delta^3)$$

with an accuracy of $O(\delta^3)$, where $O(\delta^3)$ stands for a fourth-order homogeneous polynomial in $p$ with coefficients of the form $O(\delta^3)$. Substituting the expressions found above, namely, $g_{0,1} = A_0^{-1} a$, $g_{1,1} = DA_0^{-1} b$, and $g_{2,1} = 2C_0^2(x) D^2 (A_0^{-1})^2 b$, we finally obtain

$$L^{(4)} = \delta^2 (p^4 \langle bA_0^{-1} b\rangle_{\mathbb{T}} + 4p^2 C_0^2 \langle (DA_0^{-1} b)^2\rangle_{\mathbb{T}} + 4C_0^4 \langle (D^2 (A_0^{-1})^2 b) D^2 A_0^{-1} b\rangle_{\mathbb{T}}) + O(\delta^3), \tag{4.13}$$

where $O(\delta^3)$ stands for a fourth-order homogeneous polynomial in $p$ with the coefficients of the form $O(\delta^3)$. Thus, formulas (4.11) and (4.13) enable one to calculate the polynomials $L^{(2)}(x,p)$ and $L^{(4)}(x,p)$ (in the variables $p$) approximately, with an accuracy up to $O(\delta^3)$.

One can represent formula (4.13) in a more compact form using the symmetry of $A_0^{-1}$ and $D^2$ and the antisymmetry of $D$. As was already noted above, the operators $D$ and $A_0^{-1}$ commute, and thus products of these operators can be written out in an arbitrary order of the factors. Transferring all operators in the summands of formula (4.13) to one of the factors, we can represent the formula as follows:

$$L^{(4)}(x,p) = \delta^2 \langle aA_0^{-1} (p^2 - C_0^2 D^2 A_0^{-1})^2 a\rangle_{\mathbb{T}} + O(\delta^3). \tag{4.14}$$

Restricting ourselves to the summands $\sim \delta^2$, we can express$L^{(2)}(x,p)$ and $L^{(4)}(x,p)$ in terms of the Fourier coefficients of the function $a$ in a rather simple form.

**Proposition 4.** *Let*

$$a(y,x) = \sum_{\nu \neq 0} a_\nu(x) \exp\bigl(i\langle \nu, y\rangle\bigr). \tag{4.15}$$

*Then*

$$L^{(2)}(x,p) = p^2 C_0^2 - \frac{\delta^2}{C_0^2} \sum_{\nu \neq 0} \frac{\langle p, \Theta_x \nu\rangle^2}{\langle \Theta_x \nu, \Theta_x \nu\rangle} |a_\nu|^2 + O(\delta^3), \tag{4.16}$$

$$L^{(4)}(x,p) = -\frac{\delta^2}{C_0^2} \sum_{\nu \neq 0} \frac{1}{\langle \Theta_x \nu, \Theta_x \nu\rangle} \Bigl(p^2 - 2\frac{\langle p, \Theta_x \nu\rangle^2}{\langle \Theta_x \nu, \Theta_x \nu\rangle}\Bigr)^2 |a_\nu|^2 + O(\delta^3). \tag{4.17}$$

**Proof.** Let us prove formula (4.17). Using formula (4.10), we see that

$$A_0^{-1}(p^2 - C_0^2 D^2 A_0^{-1})^2 a = -\sum_{\nu \neq 0} \frac{a_\nu}{C_0^2 \langle \Theta_x \nu, \Theta_x \nu\rangle} \Bigl(p^2 - 2\frac{\langle p, \Theta_x \nu\rangle^2}{\langle \Theta_x \nu, \Theta_x \nu\rangle}\Bigr)^2 \exp\bigl(i\langle \nu, y\rangle\bigr).$$

Since the mean of $\exp\bigl(i\langle \nu, y\rangle\bigr)$ is equal to 0 for $\nu \neq 0$ and 1 otherwise, we obtain the desired formula.

If $b$ is a finite sum of the form (4.15), then formulas (4.16) and (4.17) also have finitely many summands, and therefore the evaluation of the right-hand sides of these formulas reduces in these cases to finitely many purely algebraic operations with the Fourier coefficients of $b$ and the entries of the matrix $\Theta_x$.

Formulas (4.16) and (4.17) can be simplified in the single-phase case. Assuming that $\Theta_x$ is a vector, $y$ is a one-dimensional variable, and $\nu$ is a one-dimensional index, one can readily show that, in this case,

$$L^{(2)}(x,p) = p^2C_0^2 - \delta^2\frac{\langle p,\Theta_x\rangle^2}{\Theta_x^2C_0^2}\langle a^2\rangle_{\mathbb{T}} + O(\delta^3) \equiv p^2C_0^2 - \delta^2\frac{\langle p,\Theta_x\rangle^2}{\Theta_x^2C_0^2}\sum_{\nu\neq 0}|a_\nu|^2 + O(\delta^3). \qquad (4.18)$$

$$L^{(4)}(x,p) = -\frac{\delta^2}{\Theta_x^2C_0^2}\left(p^2 - 2\frac{\langle p,\Theta_x\rangle^2}{\Theta_x^2}\right)^2\langle(\hat{I}a)^2\rangle_{\mathbb{T}} + O(\delta^3) \equiv -\frac{\delta^2}{\Theta_x^2C_0^2}\left(p^2 - 2\frac{\langle p,\Theta_x\rangle^2}{\Theta_x^2}\right)^2\sum_{\nu\neq 0}\frac{|a_\nu|^2}{\nu^2} + O(\delta^3). \qquad (4.19)$$

**Remark 1.** Represent the formulas in Proposition 2 as follows. Treat the symbol $\lim_{\mu\to+0} w$ as the weak limit of the function $w$ with rapid oscillations. Then $C_0^2 = \lim_{\mu\to+0} C^2$ and

$$q(x,\mu) = C^2 - C_0^2 \equiv \tilde{a}(\frac{\Theta(x)}{\mu}, x). \qquad (4.20)$$

Then formulas (4.16) and (4.17) can formally be represented in the form

$$L^{(2)}(x,p) = p^2C_0^2 - \frac{1}{C_0^2}\lim_{\mu\to+0}(q\cdot\frac{\langle p,\nabla_x\rangle^2}{\triangle_x}q) + O(q^3), \qquad (4.21)$$

$$L^{(4)}(x,p) = \frac{1}{C_0^2}\lim_{\mu\to+0}\Big(\frac{1}{\mu^2\triangle_x}(p^2 - 2\frac{\langle p,\nabla_x\rangle^2}{\triangle_x})q\cdot\Big(p^2 - 2\frac{\langle p,\nabla_x\rangle^2}{\triangle_x})q\Big) + O(q^3). \qquad (4.22)$$

One can try to use these formulas in practical problems of physics and mechanics. They appeal only to the speed $C^2$ and do not use the function $\tilde{a}$, the phase $\Theta$, and so on, which are in fact not known, in contrast to $C^2$. The problem is to define the Laplace operator $\triangle_x$ correctly and to adequately calculate the weak limit (i.e., to make a correct actual averaging).

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