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Transport Properties of Two-Arc Aharonov–Bohm Interferometers with Scattering Centers

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Abstract. A model for a broad class of Aharonov–Bohm interferometers consisting of two arcs with and without scattering centers is constructed. Explicit expressions and asymptotic relations are found for the transmission coefficient for electrons in the simplest interferometers of diverse geometry (a symmetric interferometer with scattering admixture and an Aharonov–Bohm ring with two conductors attached at a single point). The influence of the relationship between the sizes of arcs and the arrangement of potentials of scattering centers, the magnetic field flux, and the energy of electrons on transport properties of the suggested nanodevices is studied.

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The solid-state interferometers using wave properties of conduction electrons are formed of mesoscopically narrow waveguides. Due to the Aharonov–Bohm effect, a modification of the magnetic flux between two branches of an interferometer of this kind leads to a periodic dependence of the conductivity caused by the interference of electron wave functions. The construction of an interferometer is usually a ring with conductors attached to it at opposite sides [1]. At present, a broad circle of investigations was also presented in Aharonov–Bohm quantum interferometers of other constructions (see, e.g., [1–5]).

In the present note, we discuss a one-dimensional model of interferometers of more general geometry in the form of segments of two rings with two wires taken at the points of splice of the rings, see Fig. 1. On these wires, the Hamiltonian can be represented in the form $H_W = \frac{1}{2m^*}\hat{p}_W^2$, where $\hat{p}_W = -i\hbar\, d/dx$ and m^* is the effective mass of the electron. In this case, the solutions of the Schrödinger equation $H_W w = Ew$ are equal to $w_1(x) = \exp(ikx)$ at the "output" W_1 and $w_2(x) = \varkappa^+\exp(ikx) + \varkappa^-\exp(-ikx)$ at the "input" W_2, where the wave number k is equal to $\sqrt{2m^*E}/\hbar$.

In the presence of an admixture at the point ψ with δ-potential at the scattering center V (cf. [2,5]), the Hamiltonian on an arc of radius R becomes

$$\widehat{H} = \widehat{H}_0 + V\delta(\varphi - \psi)\,, \qquad \widehat{H}_0 = \hat{p}^2/(2m^*R^2)\,, \tag{1}$$

where $\hat{p}$ is the operator of generalized momentum $\hbar(-id/d\varphi + \Phi)$, $\Phi = \pi R^2 B/\Phi_0$, where B is the magnetic field, $\Phi_0 = 2\pi\hbar c/e$ is the quantum of magnetic flux, and V characterizes the value of the influence of the admixture. The general Schrödinger equation $\widehat{H}f = Ef$ for the wave function f is

$$f(\varphi) = \begin{cases} \lambda^+\exp(iq^+\varphi) + \lambda^-\exp(iq^-\varphi) & \text{if} \quad 0 \leqslant \varphi < \psi\,, \\ \mu^+\exp(iq^+\varphi) + \mu^-\exp(iq^-\varphi) & \text{if} \quad \psi \leqslant \varphi < \varphi_1\,, \end{cases}$$

where $q = Rk$ and $q^\pm = \pm q - \Phi$. Using the boundary conditions $f(\psi + 0) = f(\psi - 0) \equiv f(\psi)$ and $f'(\psi + 0) - f'(\psi - 0) = vf(\psi)$, where $v = 2mVR^2/\hbar^2$, one can readily obtain equations for the coefficients $\lambda^\pm$ and $\mu^\pm$:

$$\lambda^+\theta^+ + \lambda^-\theta^- = \mu^+\theta^+ + \mu^-\theta^- \quad \text{and} \quad \mu^+\theta^+ - \mu^-\theta^- - \lambda^+\theta^+ + \lambda^-\theta^- = u(\lambda^+\theta^+ + \lambda^-\theta^-).$$

† Deceased on April 2, 2007.

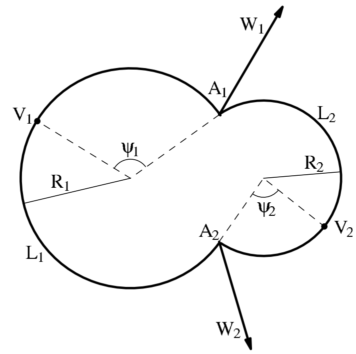

Fig. 1. Schematic picture of nanodevice formed by two arcs and two wires. On the arc L_j, the angular variable φ ranges over the segment $[0, \varphi_j]$. On the wire W_j, the variable x ranges over the values on the semiaxis $[0, \infty)$; (R_1, ψ_1) and (R_2, ψ_2) are the coordinates of the admixture centers.

Here $u = -iv/kR$ and $\theta^{\pm} = \exp(\pm iq\psi)$. At the connection points A_1 and A_2 (Fig. 1), using the Kirchhoff laws, we obtain $w_1(0) = f_1(0) = f_2(\varphi_2)$, $p_d w_1(0) + p_1 f_1(0) = p_2 f_2(\varphi_2)$ and $w_2(0) = f_2(0) = f_1(\varphi_1)$, $p_d w_2(0) + p_2 f_2(0) = p_1 f_1(\varphi_1)$. After simple but awkward manipulations, we obtain the following expression for the amplitude transmission coefficient $t_{2\to 1} = 1/\varkappa^-$:

$$t_{2\to 1} = \big(P + u_1 Q_1 + u_2 Q_2\big)\big/\big(K + u_1 L_1 + u_2 L_2 + u_1 u_2 M\big)\,, \qquad (2)$$

where

$$Q_1 = 2e^{-i\beta_2}\sin\omega_1\sin(\omega_1-\alpha_1), \qquad P = 2i(e^{-i\beta_2}\sin\alpha_1 + e^{i\beta_1}\sin\alpha_2),$$

$$Q_2 = 2e^{i\beta_1}\sin\omega_2\sin(\omega_2-\alpha_2), \qquad K = 2\cos(\beta_1+\beta_2) + \sin\alpha_1\sin\alpha_2 - 2e^{-i(\alpha_1+\alpha_2)},$$

$$L_1 = e^{-i(\alpha_1+\alpha_2)} - \tfrac{i}{2}\cos\alpha_1\sin\alpha_2 - (\cos\alpha_2 - \tfrac{i}{2}\sin\alpha_2)\cos(\alpha_1 - 2\omega_1),$$

$$L_2 = e^{-i(\alpha_1+\alpha_2)} - \tfrac{i}{2}\cos\alpha_2\sin\alpha_1 - (\cos\alpha_1 - \tfrac{i}{2}\sin\alpha_1)\cos(\alpha_2 - 2\omega_2),$$

$$M = \tfrac{1}{4}\big[\cos(\alpha_1-2\omega_1)\cos(\alpha_2-2\omega_2) + 2\sin(\alpha_1-2\omega_1)\sin(\alpha_2-2\omega_2) - 2e^{-i(\alpha_1+\alpha_2)} - \cos\alpha_1\cos\alpha_2 + (\cos\alpha_2 - 2i\sin\alpha_2)\cos(\alpha_1-2\omega_1) + (\cos\alpha_1 - 2i\sin\alpha_1)\cos(\alpha_2-2\omega_2)\big].$$

Here we have introduced the notation $\alpha_j = kR_j\varphi_j$, $\beta_j = \Phi_j\varphi_j$, and $\omega_j = kR_j\psi_j$. The probability that the electrons pass through the device in question is determined by the coefficient $T = |t_{2\to 1}|^2$.

In this note we consider the simplest cases of the scheme suggested above for the Aharonov–Bohm interferometer. For an ordinary two-terminal ring ($R_{1,2} = R$, $\varphi_{1,2} = \pi$) without admixture centers (and thus $u_1 = u_2 = 0$), we obtain the known expression [2]

$$t = i\sin(kL)\cos(\pi\Phi)\big/\big(5\sin^2(kL)/4 - \sin^2(\pi\Phi) + i\sin(2kL)/2\big)\,,$$

where $L = \pi R$ is a half of the ring. The graph of the transmission coefficient T (the black field corresponds to $T = 0$) in dependence on the normalized magnetic flux Φ and the electron energy E/E_0 (here $E_0 = E_1(\Phi = 0) = \hbar^2/(2m^*R^2)$ is the first eigenvalue of an isolated ring in the absence of magnetic field) is represented in Fig. 2a. Here the curves indicate the eigenvalues of the energy $E_n(\Phi) = (n\pm\Phi)^2 E_0$ for an isolated Aharonov–Bohm ring. An asymmetry of "shoulders" of the ring, which is usual in experiments, leads to a distortion of the picture $T(\Phi)$, which can be seen by the example in Fig. 2b with the ring composed of the arcs $\varphi_1 = 0.85\pi$ and $\varphi_1 = 1.15\pi$ (to obtain this graph, we have used expression (2)). As the energy increases, a phase shift along the flow is observed. For instance, the nontransparency domains for integral flows $\Phi = 0, \pm 1, \pm 2, \dots$ and the energies $E \approx 12E_0$ are replaced by nontransparency domains for half-integer flows $\Phi = \pm 0.5, \pm 1.5, \dots$ for energies $E \approx 50E_0$.

An essential modification of the behavior of the transmission coefficient $T(E, \Phi)$ occurs if one of the "shoulders" contains a scattering center. Even for a symmetric interferometer, entire nongating domains occur (on which $T \simeq 0$) for certain electron energies (i.e., when supplying the ends of the wires with certain electric voltage [1–5]), and domains of almost full passing on which $T \simeq 1$. This can be clearly seen in the example of an interferometer with scattering center $V = 16E_0$ placed at the point $\psi = \pi/2$, see Fig. 3. Here the domains of gating belong to the intervals of energy from 0.3 to 2.4 and from 6.0 to 12.0 and the domains of shutting are placed near 3.0, 4.5, 14.0, and 17.0.

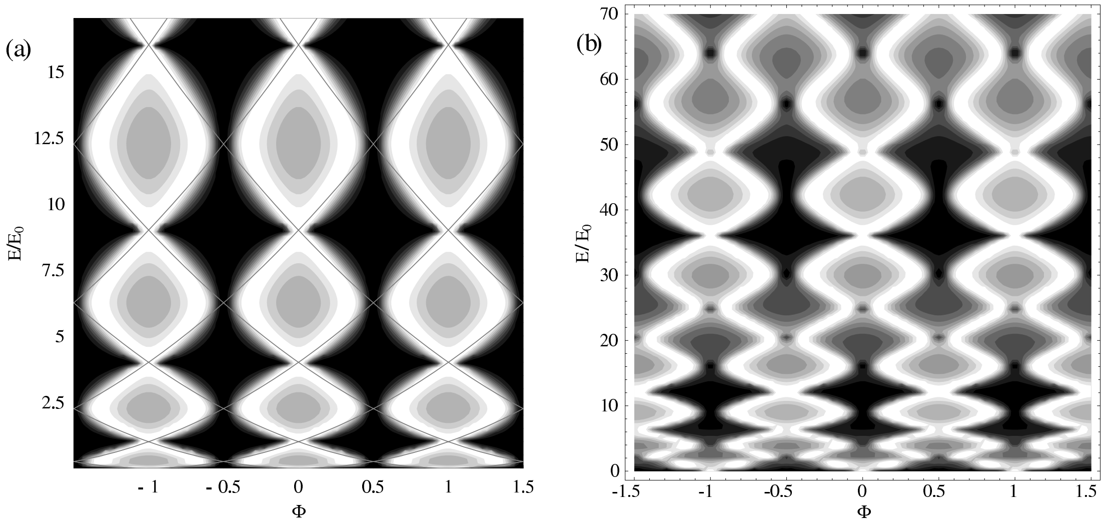

Fig. 2. Dependencies of the transmission coefficient $T(E,\Phi)$ for a ballistic ring without admixtures: a) a symmetric case, equal "shoulders" $(\varphi_{1,2}=\pi)$; b) an asymmetric case, the "shoulders" of somewhat different length with $\varphi_1=0.85\pi$ and $\varphi_2=1.15\pi$.

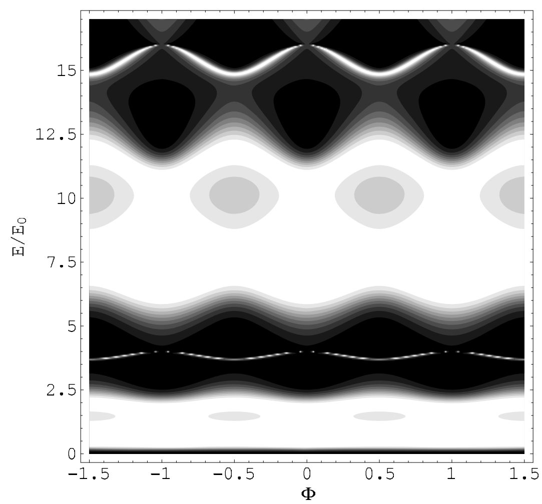

Fig. 3. The dependence of the transmission coefficient $T(E,\Phi)$ for a symmetric interferometer with scattering center $V=16E_0$ at the point $\psi=\pi/2$.

The presence of two centers in both the "shoulders" can lead to narrow doubled resonances (with maximal values) in the transmission coefficient $T(E,\Phi)$, as shown in Fig. 4. In the case of a symmetric ring $(\varphi_{1,2}=\pi)$, for chosen values of the potentials of the scattering centers placed at the points $\psi_{1,2}=\pi/2$, in the interval of sufficiently small values of the magnetic flux, there are sharp "spines" of the maximal transmission coefficient at small electron energies and a "plateau" for large energies.

Aharonov–Bohm ring connected to two wires at a single point.

The model constructed above enables one to consider the case of Aharonov–Bohm ring ($R_1=R$, $\varphi_1=2\pi$, and $\varphi_2=0$) connected to two conducting channels at a single point; see Fig. 5.

Let us place a scattering potential $V_1=V$ at an arbitrary but chosen point $\psi_1=\psi$. In this case, the transmission coefficient can be represented in the form

$$t_{2\to1}=\frac{i\sin 2kL+u\sin\psi kR\sin(\psi kR-2kL)}{\cos 2\pi\Phi-\exp(-2ikL)+u[\exp(-2ikL)-\cos(2kL-2\psi kL)]/2}, \tag{3}$$

where L is the length of the half-ring. If there is no scatterer, then the expression can be simplified

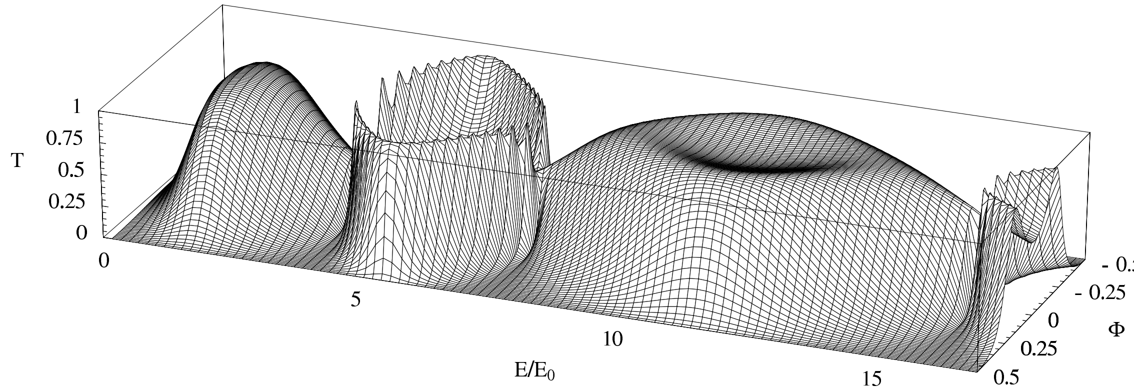

Fig. 4. Transmission coefficient for a symmetric interferometer $\varphi_{1,2} = \pi$ with two scattering centers at the points $\psi_{1,2} = \pi/2$. The values $V_{1,2} = -2.4E_0$ are chosen to obtain a broad plateau in dependence on $T(E, \Phi)$.

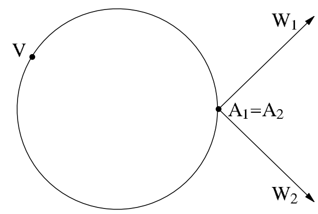

Fig. 5. Aharonov–Bohm ring with scattering center connected to channels at a single point.

and becomes

$$t_{2\to 1} = \frac{i \sin 2kL}{\cos 2\pi\Phi - \cos 2kL + i \sin 2kL}. \tag{4}$$

One can readily see that $t_{2\to1} = 1$ if $\cos 2\pi\Phi = \cos 2kL$. The solutions of this equation $kR = n \pm \Phi$ are the eigenvalues of the isolated Aharonov–Bohm ring. Thus, for a nanodevice consisting of a ballistic ring with channels taken at the same point, the probability of passage is equal to 1 if the electron energies coincide with the eigenvalues $E_n(\Phi) = (n \pm \Phi)^2 E_0$ of the isolated Aharonov–Bohm ring. Note that, in the case of channels symmetrically connected to the ring, the probability of electron passage vanishes at the eigenvalues of the isolated Aharonov–Bohm ring.

The presence of a scatterer on a ring connected to channels at a single point leads to the formation of sharp resonances (similar to admixture levels on the momentum–energy diagram) and broad resonances (similar to conductivity bands), as is the case for a ring with scattering center and diametrally attached wires-channels. If the scattering center is placed at the point $\psi = m\pi/n$ (where m and n are coprime positive integers), then sharp resonances occur near the energy values $E_k = (nk/m)^2 E_0$, where $k = 1, 2, \ldots$

Let us consider the symmetric case in detail: the channels are attached to the ring at the point $\varphi = 0$, and the scattering center is placed at the point $\psi = \pi$. Formula (3) becomes

$$t_{2\to 1} = \frac{i(\cos kL + \tilde{u} \sin kL) \sin kL}{\sin^2 \pi\Phi - \sin^2 kL + \tilde{u} \sin 2kL + i(\cos kL + \tilde{u} \sin kL) \sin kL}, \tag{5}$$

where $\tilde{u} = iu/2 = V/2\sqrt{E_0 E}$. One can readily see that the transmission coefficient vanishes if $\cos kL + \tilde{u} \sin kL = 0$ or $\sin kL = 0$, i.e., for the electrons with the energies

$$E_n^{(l)} = E_0 \left(n - (1/\pi) \arctan(1/\tilde{u})\right)^2, \qquad E_n^{(u)} = E_0 n^2. \tag{6}$$

As the ratio $V/\sqrt{E_0 E}$ increases (i.e., the power of the scatterer increases with respect to the electron energy), the widths of the intervals $(E_n^{(l)}, E_n^{(u)})$ diminish to a greater extent for small energies (small values of the index n); see Fig. 6.

One can also readily see from formula (5) that the transmission coefficient is equal to 1 if the following equation holds:

$$\sin^2 \pi\Phi - \sin^2 kL + \tilde{u} \sin 2kL = 0. \tag{7}$$

Suppose that the condition $\frac{1}{\pi} \arctan \frac{1}{\tilde{u}} \ll 1$ (of strong scattering potential) is satisfied. In this case, near $E_n^{(u)}$, one can assume that $\sin^2 kL \approx 0$ and $\cos kL \approx (-1)^n$, and an approximate solution of

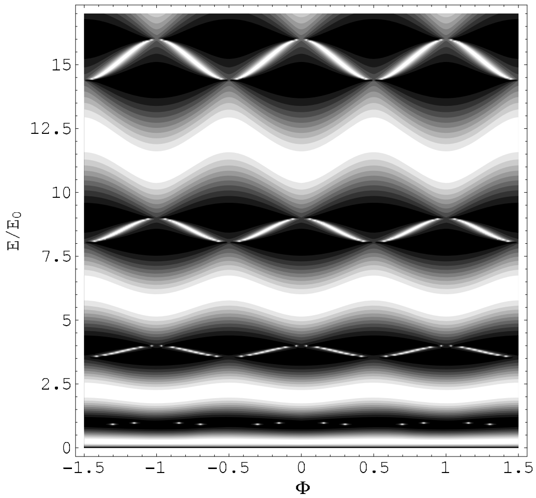

Fig. 6. Transmission coefficient of an Aharonov–Bohm ring containing a scattering center ($V = 10E_0$, $\psi = \pi$) and attached to the channels at a single point.

equation (7) becomes $\sin kL \approx \frac{(-1)^n}{\tilde{u}} \sin^2 \pi\Phi$. Thus, the position of sharp resonances is determined by the approximate formula

$$E_n^{(\mathrm{res})} \approx E_0\big(n - (1/\pi) \arcsin\big((1/\tilde{u}) \sin^2 \pi\Phi\big)\big)^2. \tag{8}$$

This formula implies that the position of a resonance inside the interval $(E_n^{(l)}, E_n^{(u)})$ depends on the magnetic field flux, Φ. As the flow of the field approaches integer multiples of the quantum of the magnetic flux, the resonances tend to $E_n^{(u)}$ and, as the flow approaches half-integral multiples of the quantum of the flow Φ, the resonances tend to the energies $E_n^{(l)}$. The resonances collapse in both the limits because the points $(m, E_n^{(u)})$ and $(m + 1/2, E_n^{(l)})$, where m is an integer, are discontinuity points of the function $t_{2\to1}(\Phi, E)$. This is an artefact of our model. In experimental conditions, washing out due to heating must "suppress" these singularities.

For the flows $\Phi = (2m + 1)/4$, where m is an integer, sharp resonances are placed at the middle of the intervals $(E_n^{(l)}, E_n^{(u)})$ and are of symmetric form (see Fig. 7).

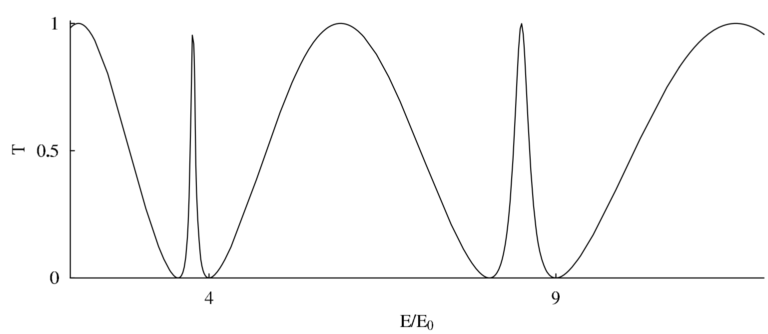

Fig. 7. Transmission coefficient of an Aharonov–Bohm ring containing a scattering center and attached to the channels at a single point, as a function of energy, has symmetric resonances of Breit–Wigner type for the flow $\Phi = 1/4$.

For magnetic field fluxes that differ from the products of numbers of the form $m/4$ and the quantum of the flow, the sharp resonances become asymmetric and somewhat recall the Fano resonances, see Fig. 8.

Let us find expansions of the trigonometric functions in expression (5) for energies E lying in neighborhoods of the points $E_n^{(u)} = n^2 E_0$ (the eigenvalues of the Aharonov–Bohm ring in the absence of magnetic field) such that these expansions are linear with respect to $E - E_n^{(u)}$. Note that

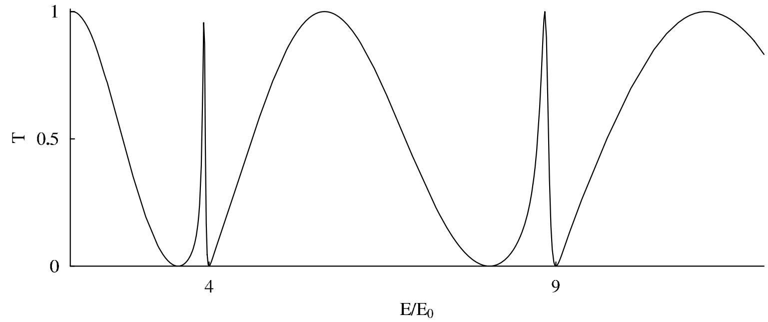

Fig. 8. Transmission coefficient of an Aharonov–Bohm ring containing a scattering center and attached to two channels at a single point. The "right" resonances are placed near the energies $E_n^{(u)}$. The flow is $\Phi = 1/8$.

$kL = \pi kR = \pi\sqrt{\frac{E}{E_0}} = \pi n\sqrt{\frac{E-E_n^{(u)}}{n^2E_0}+1}$, and we can write $kL = \pi n + \delta_n$ near $E - E_n^{(u)}$, where δ_n is small. Therefore,

$$\delta_n = \pi n\left(\sqrt{\frac{E-E_n^{(u)}}{n^2E_0}+1}-1\right) \approx \frac{\pi}{2n}\frac{E-E_n^{(u)}}{E_0}, \quad \sin kL \approx (-1)^n\frac{\pi}{2n}\frac{E-E_n^{(u)}}{E_0}, \quad \cos kL \approx (-1)^n.$$

Substituting the last expression into formula (5) and omitting the subsequent terms small with respect to δ_n, we finally obtain

$$t_{2\to 1} \approx \frac{i\pi\left(E-E_n^{(l)}\right)\left(E-E_n^{(u)}\right)/2nE_0}{E-E_n^{(\mathrm{res})}+i\pi\left(E-E_n^{(l)}\right)\left(E-E_n^{(u)}\right)/2nE_0}. \tag{9}$$

One can readily see that the transmission coefficient evaluated by the approximate formula (9) has the same behavior near resonances as the coefficient evaluated by formula (5), namely, it vanishes at $E = E_n^{(l)}$ and $E = E_n^{(u)}$ and is equal to one at $E = E_n^{(\mathrm{res})}$. The resonance is of symmetric shape, see Fig 7, if it is placed in the middle between $E_n^{(l)}$ and $E_n^{(u)}$, i.e., if $\sin^2\pi\Phi = 1/2$ (the magnetic field flux is equal to $(2m+1)/4$ for some integer m) and asymmetric shape, similar to Fano resonances, otherwise; see Fig. 8.

Thus, for a quasi-one-dimensional model, having obtained a general expression for $T(E,\Phi)$, we have carried out the investigations of the simplest cases and showed that the behavior of the transmission coefficient substantially differs from the simple case and from configurations of the Aharonov–Bohm interferometers that have been studied earlier [1–5]. Note that the width of the narrow resonances for the simple structure of the Aharonov–Bohm ring containing a scattering center and attached to two channels at a single point heavily depends on the magnetic field. These resonances can be of Fano type or of Breit–Wigner type or can be completely absent at the eigenvalues of the isolated ring.

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REFERENCES

1. T. Ihn, A. Fuhrer, L. Meier, M. Sigrist, and K. Ensslin, "Quantum Physics in Quantum Rings," Europhys. News **38** (3), 78–81 (2005).
2. V. A. Geyler, V. V. Demidov, and V. A. Margulis, "Transport in the Two-Terminal Aharonov–Bohm Ring," Zh. Tekhn. Fiz. **73** (6), 1–8 (2003).
3. M. Grochol, F. Grosse, and R. Zimmermann, "Optical Exciton Aharonov–Bohm Effect, Persistent Current, and Magnetization in Semiconductor Nanorings of Type I and II," Phys. Rev. B **74**, 115416 (2006).
4. R. Leturcq, D. Sánchez, G. Götz, T. Ihn, K. Ensslin, D. C. Driscoll, and A. C. Gossard, "Magnetic Field Symmetry and Phase Rigidity of the Nonlinear Conductance in a Ring," Phys. Rev. Lett. **96** 126801 (2006).
5. P. Vasilopoulos, O. Kálmán, F. M. Peeters, and M. G. Benedict, "Aharonov–Bohm Oscillations in a Mesoscopic Ring with Asymmetric Arm-Dependent Injection," Phys. Rev. B **75**, 035304 (2007).