INDEX THEORY FOR BASIC DIRAC OPERATORS ON RIEMANNIAN FOLIATIONS

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Abstract. In this paper we prove a formula for the analytic index of a basic Dirac-type operator on a Riemannian foliation, solving a problem that has been open for many years. We also consider more general indices given by twisting the basic Dirac operator by a representation of the orthogonal group. The formula is a sum of integrals over blowups of the strata of the foliation and also involves eta invariants of associated elliptic operators. As a special case, a Gauss-Bonnet formula for the basic Euler characteristic is obtained using two independent proofs.

Contents

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1. Introduction

Let $(M,\mathcal{F})$ be a smooth, closed manifold endowed with a Riemannian foliation. Let $D_b^E\colon \Gamma_b\left(M,E^+\right)\to\Gamma_b\left(M,E^-\right)$ be a basic, transversally elliptic differential operator acting on the basic sections of a foliated vector bundle E. The basic index $\operatorname{ind}_b\left(D_b^E\right)$ is known to be a well-defined integer, and it has been an open problem since the 1980s to write this integer in terms of geometric and topological invariants. Our main theorem (Theorem 6.1) expresses $\operatorname{ind}_b\left(D_b^E\right)$ as a sum of integrals over the different strata of the Riemannian foliation, and it involves the eta invariant of associated equivariant elliptic operators on spheres normal to the strata. The result is

$$\operatorname{ind}_b\left(D_b^E\right)=\int_{\widetilde{M_0\diagup\mathcal{F}}}A_{0,b}\left(x\right)\ \widetilde{\left|dx\right|}+\sum_{j=1}^{r}\beta\left(M_j\right)\ ,$$
$$\beta\left(M_j\right)=\frac{1}{2}\sum_{\tau}\frac{1}{n_\tau\operatorname{rank}\,W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right)+h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j\diagup\mathcal{F}}}A_{j,b}^{\tau}\left(x\right)\ \widetilde{\left|dx\right|}\ .$$

The notation will be explained later; the integrands $A_{0,b}\left(x\right)$ and $A_{j,b}^{\tau}\left(x\right)$ are the familar Atiyah-Singer integrands corresponding to local heat kernel supertraces of induced elliptic operators over closed manifolds. Even in the case when the operator D is elliptic, this result was not known previously. We emphasize that every part of the formula is explicitly computable from local information provided by the operator and foliation. Even the eta invariant of the operator $D_j^{S+,\tau}$ on a sphere is calculated directly from the principal transverse symbol of the operator D_b^E at one point of a singular stratum. The de Rham operator provides an important example illustrating the computability of the formula, yielding the basic Gauss-Bonnet Theorem (Theorem 10.1).

This new theorem is proved by first writing $\operatorname{ind}_b\left(D_b^E\right)$ as the invariant index of a G-equivariant, transversally elliptic operator $\mathcal{D}$ on a G-manifold $\widehat{W}$ associated to the foliation, where G is a compact Lie group of isometries. Using our equivariant index theorem in [14], we obtain an expression for this index in terms of the geometry and topology of $\widehat{W}$ and then rewrite this formula in terms of the original data on the foliation.

We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations in a very special case. Using a different technique, they prove a formula for the index of a basic Dirac operator that is distinct from our formula, in the case where all the infinitesimal holonomy groups of the foliation are connected tori and if Molino's commuting sheaf is abelian and has trivial holonomy (see [27]). Our result requires at most mild topological assumptions on the transverse structure of the strata of the Riemannian foliation. In particular, the Gauss-Bonnet Theorem for Riemannian foliations (Theorem 10.1) is a corollary and requires no assumptions on the structure of the Riemannian foliation.

The paper is organized as follows. The definitions of the basic sections, holonomy-equivariant vector bundles, basic Clifford bundles, and basic Dirac-type operators are given in Section 2. In Section 3, we describe the Fredholm properties of the basic index and show how to construct the G-manifold $\widehat{W}$ and the G-equivariant operator $\mathcal{D}$, using a generalization of Molino theory [48]. We also use our construction to obtain asymptotic expansions and eigenvalue asymptotics of transversally elliptic operators on Riemannian foliations in Section 3.2, which is of independent interest. In Section 3.4, we construct bundles associated to represention of the isotropy subgroups of the G-action; these bundles are used in the main theorem. In Section 4, we describe a method of cutting out tubular neighborhoods of the singular strata of the foliation and doubling the remainder to produce a Riemannian foliation with fewer strata. We also deform the operator and metric and determine the effect of this desingularization operation on the basic index. We recall the equivariant index theorem in [14] in Section 5 and prove the basic index theorem in Section 6. Finally, we prove a generalization of this theorem to representation-valued basic indices in Section 7.

We illustrate the theorem with a collection of examples. These include foliations by suspension (Section 8), a transverse signature (Section 9), and the basic Gauss-Bonnet Theorem (Section 10).

One known application of our theorem is Kawasaki's Orbifold Index Theorem ([39], [40]). It is known that every orbifold is the leaf space of a Riemannian foliation, where the leaves are orbits of an orthogonal group action such that all isotropy subgroups have the same dimension. In particular, the contributions from the eta invariants in our transverse signature example (Section 9) agree exactly with the contributions from the singular orbifold strata when the orbifold is four-dimensional.

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2. Riemannian foliations and basic Dirac operators

2.1. **Basic definitions.** A foliation of codimension q on a smooth manifold M of dimension n is a natural generalization of a submersion. Any submersion $f\colon M \to N$ with fiber dimension p induces locally, on an open set $U \subset M$, a diffeomorphism $\phi\colon U \to \mathbb{R}^q \times \mathbb{R}^p \ni (y,x)$, where $p+q=n$. A **foliation** $\mathcal{F}$ is a (maximal) atlas $\{\phi_\alpha\colon U_\alpha \to \mathbb{R}^q \times \mathbb{R}^p\}$ of M such that the transition functions $\phi_\alpha \circ \phi_\beta^{-1}\colon \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q \times \mathbb{R}^p$ preserve the fibers, i.e. they have the form

$$\phi_\alpha \circ \phi_\beta^{-1}(y,x) = (\tau_{\alpha\beta}(y), \psi_{\alpha\beta}(x,y)).$$

This local description has many equivalent formulations, as expressed in the famous Frobenius Theorem. Geometrically speaking, M is partitioned into p-dimensional immersed submanifolds called the **leaves** of the foliation; the tangent bundle $T\mathcal{F}$ to the leaves forms an integrable subbundle of the tangent bundle TM.

In the case of a submersion, the normal bundle to $T\mathcal{F}$ is naturally identified with the tangent bundle of the base, which then forms the space of leaves. In general, such a description is not possible, since the space of leaves defined by the obvious equivalence relation does not form a manifold. Nevertheless, reasonable **transverse geometry** can be expressed in terms of the normal bundle $Q := TM \diagup T\mathcal{F}$ of the foliation. We are particularly interested in the case of a Riemannian foliation, which generalizes the concept of a Riemannian submersion. That is, the horizontal metric g_h on the total space of a Riemannian submersion is the pullback of the metric on the base, such that in any chart ϕ as above, $g_h\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_i}\right)$ depends on the base coordinates y alone. Another way to express this is that $\mathcal{L}_X g_h = 0$ for all vertical vector fields X, where $\mathcal{L}_X$ denotes the Lie derivative. In the case of a foliation, the normal bundle Q is framed by $\left\{\frac{\partial}{\partial y_j}\right\}_{j=1}^{q}$, and this foliation is called **Riemannian** if it is equipped with a metric g_Q on Q such that $\mathcal{L}_X g_Q = 0$ for all $X \in C^\infty(M, T\mathcal{F})$ (see [48], [58]). For example, a Riemannian foliation with all leaves compact is a (generalized) Seifert fibration; in this case the leaf space is an orbifold ([48, Section 3.6]). Or if a Lie group of isometries of a Riemannian manifold has orbits of constant dimension, then the orbits form a Riemannian foliation. A large class of examples of Riemannian foliations is produced by suspension (see Section 8).

Consider the exact sequence of vector bundles

$$0 \to T\mathcal{F} \to TM \xrightarrow{\pi} Q \to 0.$$

The **Bott connection** ∇^Q on the normal bundle Q is defined as follows. If $s \in C^\infty(Q)$ and if $\pi(Y) = s$, then $\nabla^Q_X s = \pi([X, Y])$. The basic sections of Q are represented by **basic vector fields**, fields whose flows preserve the foliation. Alternately, a section V of Q is called a basic vector field if for every $X \in C^\infty(T\mathcal{F})$, $[X, V] \in C^\infty(T\mathcal{F})$ (see [33] or [48]).

A differential form ω on M is **basic** if locally it is a pullback of a form on the base. Equivalently, ω is basic if for every vector field X tangent to the leaves, $i_X\omega = 0$ and $i_X(d\omega) = 0$, where i_X denotes interior product with X. If we extend the Bott connection to a connection $\nabla^{\Lambda^* Q^*}$ on $\Lambda^* Q^*$, a section ω of $\Lambda^* Q^*$ is basic if and only if $\nabla^{\Lambda^* Q^*}_X \omega = 0$ for all X tangent to $\mathcal{F}$. The exterior derivative of a basic form is again basic, so the basic forms are a subcomplex $\Omega^*(M, \mathcal{F})$ of the de Rham complex $\Omega^*(M)$. The cohomology of this subcomplex is the **basic cohomology** $H^*(M, \mathcal{F})$.

2.2. **Foliated vector bundles.** We now review some standard definitions (see [33] and [48]). Let G be a compact Lie group. With notation as above, we say that a principal G–bundle $P \to (M, \mathcal{F})$ is a **foliated principal bundle** if it is equipped with a foliation $\mathcal{F}_P$ (the **lifted foliation**) such that the distribution $T\mathcal{F}_P$ is invariant under the right action of G, is transversal to the tangent space to the fiber, and projects to $T\mathcal{F}$. A connection ω on P is called **adapted to** $\mathcal{F}_P$ if the associated horizontal distribution contains $T\mathcal{F}_P$. An adapted connection ω is called a **basic connection** if it is basic as a $\mathfrak{g}$-valued form on $(P, \mathcal{F}_P)$. Note that in [33] the authors showed that basic connections always exist on a foliated principal bundle over a Riemannian foliation.

Similarly, a vector bundle $E \to (M, \mathcal{F})$ is **foliated** if E is associated to a foliated principal bundle $P \to (M, \mathcal{F})$ via a representation ρ from G to $O(k)$ or $U(k)$. Let $\Omega(M, E)$ denote the space of forms on M with coefficients in E. If a connection form ω on P is adapted, then we say that an associated covariant derivative operator ∇^E on $\Omega(M, E)$ is **adapted** to the foliated bundle. We say that ∇^E is a **basic** connection on E if in addition the associated curvature operator $\left(\nabla^E\right)^2$ satisfies $i_X\left(\nabla^E\right)^2 = 0$ for every $X \in T\mathcal{F}$, where i_X denotes the interior product with X. Note that ∇^E is basic if ω is basic.

Let $C^\infty(E)$ denote the smooth sections of E, and let ∇^E denote a basic connection on E. We say that a section $s\colon M \to E$ is a **basic section** if and only if $\nabla^E_X s = 0$ for all $X \in T\mathcal{F}$. Let $C^\infty_b(E)$ denote the space of basic sections of E. We will make use of the fact that we can give E a metric such that ∇^E is a metric basic connection.

The **holonomy groupoid** $G_\mathcal{F}$ of $(M, \mathcal{F})$ (see [59]) is the set of ordered triples $(x, y, [\gamma])$, where x and y are points of a leaf L and $[\gamma]$ is an equivalence class of piecewise smooth paths in L starting at x and ending at y; two such paths α and β are equivalent if and only if $\beta^{-1}\alpha$ has trivial holonomy. Multiplication is defined by $(y, z, [\alpha]) \cdot (x, y, [\beta]) = (x, z, [\alpha\beta])$, where

$\alpha\beta$ refers to the curve starting at x and ending at z that is the concatenation of β and α. Because $(M,\mathcal{F})$ is Riemannian, $G_{\mathcal{F}}$ is endowed with the structure of a smooth $(n+p)$–dimensional manifold (see [59]), where n is the dimension of M and p is the dimension of the foliation.

We say that a vector bundle $E\rightarrow M$ is $G_{\mathcal{F}}$**–equivariant** if there is an action of the holonomy groupoid on the fibers. Explicitly, if the action of $g=(x,y,[\gamma])$ is denoted by T_g, then $T_g\colon E_x\rightarrow E_y$ is a linear transformation. The transformations $\{T_g\}$ satisfy $T_gT_h=T_{g\cdot h}$ for every $g,h\in G_{\mathcal{F}}$ for which $g\cdot h$ is defined, and we require that the map $g\longmapsto T_g$ is smooth. In addition, we require that for any unit $u=(x,x,[\alpha])$ (that is, such that the holonomy of α is trivial), $T_u\colon E_x\rightarrow E_x$ is the identity.

We say that a section $s\colon M\rightarrow E$ is **holonomy–invariant** if for every $g=(x,y,[\gamma])\in G_{\mathcal{F}}$, $T_gs\left(x\right)=s\left(y\right)$.

Remark 2.1. *Every $G_{\mathcal{F}}$–equivariant vector bundle $E\rightarrow(M,\mathcal{F})$ is a foliated vector bundle, because the action of the holonomy groupoid corresponds exactly to parallel translation along the leaves. If the partial connection is extended to a basic connection on E, we see that the notions of basic sections and holonomy–invariant sections are the same.*

On the other hand, suppose that $E\rightarrow(M,\mathcal{F})$ is a foliated vector bundle that is equipped with a basic connection. It is not necessarily true that parallel translation can be used to give E the structure of a $G_{\mathcal{F}}$–equivariant vector bundle. For example, let α be an irrational multiple of 2π, and consider $E=[0,2\pi]\times[0,2\pi]\times\mathbb{C}\diagup(0,\theta,z)\sim(2\pi,\theta,e^{i\alpha}z)$, which is a Hermitian line bundle over the torus $S^1\times S^1$, using the obvious product metric. The natural flat connection for E over the torus is a basic connection for the product foliation $\mathcal{F}=\{L_\theta\}$, where $L_\theta=\{(\phi,\theta)\mid\phi\in S^1\}$. However, one can check that parallel translation cannot be used to make a well-defined action of $G_{\mathcal{F}}$ on the fibers.

An example of a $G_{\mathcal{F}}$–equivariant vector bundle is the normal bundle Q, given by the exact sequence of vector bundles

$$0\rightarrow T\mathcal{F}\rightarrow TM\xrightarrow{\pi}Q\rightarrow 0.$$

The Bott connection ∇^Q on Q is a metric basic connection. (Recall that if $s\in C^\infty\left(Q\right)$ and if $\pi\left(Y\right)=s$, then $\nabla^Q_Xs=\pi\left([X,Y]\right)$.) The basic sections of Q are represented by **basic vector fields**, fields whose flows preserve the foliation. Alternately, a section V of Q is called a basic vector field if for every $X\in C^\infty\left(T\mathcal{F}\right)$, $[X,V]\in C^\infty\left(T\mathcal{F}\right)$ (see [33] or [48]).

Lemma 2.2. *Let $E\rightarrow(M,\mathcal{F})$ be a foliated vector bundle with a basic connection ∇^E. Let $V\in C^\infty\left(Q\right)$ be a basic vector field, and let $s\colon M\rightarrow E$ be a basic section. Then ∇^E_Vs is a basic section of E.*

Proof. For any $X\in C^\infty\left(T\mathcal{F}\right)$, $[X,V]\in C^\infty\left(T\mathcal{F}\right)$, so that $\nabla^E_Xs=\nabla^E_{[X,V]}s=0$. Thus,

$$\begin{aligned}\nabla^E_X\nabla^E_Vs&=\left(\nabla^E_X\nabla^E_V-\nabla^E_V\nabla^E_X-\nabla^E_{[X,V]}\right)s\\&=\left(\nabla^E\right)^2(X,V)\,s=0,\end{aligned}$$

since ∇^E is basic. □

Another example of a foliated vector bundle is the exterior bundle $\bigwedge Q^*$; the induced connection from the Bott connection on Q is a metric basic connection. The set of basic sections of this vector bundle is the set of basic forms $\Omega\left(M,\mathcal{F}\right)$, which is defined in the ordinary way in Section 2.1. It is routine to check that these two definitions of basic forms are equivalent.

2.3. **Basic Clifford bundles.** Identifying Q with the normal bundle of the Riemannian foliation $(M,\mathcal{F})$, we form the bundle of Clifford algebras $\mathbb{C}\mathrm{l}\,(Q)=\mathrm{Cl}\,(Q)\otimes\mathbb{C}$ over M.

Definition 2.3. *Let E be a bundle of $\mathbb{C}\mathrm{l}\,(Q)$ –modules over a Riemannian foliation $(M,\mathcal{F})$. Let ∇ denote the Levi–Civita connection on M, which restricts to a metric basic connection on Q. Let $h=(\cdot,\cdot)$ be a Hermitian metric on E, and let ∇^E be a connection on E. Let the action of an element $\xi\in\mathbb{C}\mathrm{l}\,(Q_x)$ on $v\in E_x$ be denoted by $c\,(\xi)\,v$. We say that $\left(E,h,\nabla^E\right)$ is a **basic Clifford bundle** if*

1. *The bundle $E\to(M,\mathcal{F})$ is foliated.*
2. *The connection ∇^E is a metric basic connection.*
3. *For every $\xi\in Q_x$, $c\,(\xi)$ is skew-adjoint on E_x.*
4. *For every $X\in C^\infty(TM)\,,Y\in C^\infty(Q)\,,$ and $s\in C^\infty(E)\,,$*
$$\nabla_X^E\left(c\,(Y)\,s\right)=c\,(\nabla_XY)\,s+c\,(Y)\,\nabla_X^E\,(s)\,.$$

Lemma 2.4. *Let $\left(E,h,\nabla^E\right)$ be a basic Clifford module over $(M,\mathcal{F})$. Let $V\in C^\infty(Q)$ be a basic vector field, and let $s\colon M\to E$ be a basic section. Then $c\,(V)\,s$ is a basic section of E.*

Proof. If $\nabla_X^Es=0$ and $\nabla_XV=0$ for every $X\in C^\infty(T\mathcal{F})$, then
$$\nabla_X^E\left(c\,(V)\,s\right)=c\,(\nabla_XV)\;\;s+c\,(V)\,\nabla_X^E\,(s)=0.$$
□

2.4. **Basic Dirac operators.**

Definition 2.5. *Let $\left(E,(\cdot,\cdot)\,,\nabla^E\right)$ be a basic Clifford bundle. The **transversal Dirac operator** D_{tr}^E is the composition of the maps*
$$C^\infty(E)\overset{\left(\nabla^E\right)^{\mathrm{tr}}}{\to}C^\infty(Q^*\otimes E)\overset{\cong}{\to}C^\infty(Q\otimes E)\overset{c}{\to}C^\infty(E)\,,$$
where the operator $\left(\nabla^E\right)^{\mathrm{tr}}$ is the obvious projection of $\nabla^E\colon C^\infty(E)\to C^\infty(T^*M\otimes E)$ and the ismorphism $\cong$ is induced via the holonomy–invariant metric on Q.

If $\{e_1,...,e_q\}$ is an orthonormal basis of Q, we have that
$$D_{\mathrm{tr}}^E=\sum_{j=1}^{q}c\,(e_j)\,\nabla_{e_j}^E.$$

Let $p\colon T^*M\to M$ be the projection. The restriction of the principal symbol $\sigma\left(D_{\mathrm{tr}}^E\right)$: $T^*M\to\mathrm{End}\,(p^*E)$ to Q^* is denoted $\sigma^{\mathrm{tr}}\left(D_{\mathrm{tr}}^E\right)$, and it is given by
$$\sigma^{\mathrm{tr}}\left(D_{\mathrm{tr}}^E\right)(\xi)=c\left(\xi^\#\right).$$

Since this map is invertible for $\xi\in Q^*\setminus 0$, we say that D_{tr}^E is **transversally elliptic**.

Lemma 2.6. *The operator D_{tr}^E restricts to a map on the subspace $C_b^\infty(E)$.*

Proof. Suppose that $s\colon M\to E$ is a basic section, so that $\nabla_X^Es=0$ for every $X\in C^\infty(T\mathcal{F})$. Near a point x of M, choose an orthonormal frame field $(e_1,...,e_q)$ of Q consisting of basic fields. Then
$$\nabla_X^E\left(D_{\mathrm{tr}}^E\,(s)\right)=\sum_{j=1}^{q}\nabla_X^E\left(c\,(e_j)\,\nabla_{e_j}^Es\right)$$

$$= \sum_{j=1}^{q} c\left(e_j\right) \nabla_X^E \left(\nabla_{e_j}^E s\right),$$

since each e_j is basic, and the result is zero by Lemma 2.2. □

We now calculate the formal adjoint of D_{tr}^E on $C_b^\infty\left(E\right)$. Letting $\left(s_1, s_2\right)$ denote the pointwise inner product of sections of E and choosing an orthonormal frame field $\left(e_1, ..., e_q\right)$ of Q consisting of basic fields, we have that

$$\begin{aligned}
\left(D_{\mathrm{tr}}^E s_1, s_2\right) - \left(s_1, D_{\mathrm{tr}}^E s_2\right) &= \sum_{j=1}^{q} \left(c\left(e_j\right) \nabla_{e_j}^E s_1, s_2\right) - \left(s_1, c\left(e_j\right) \nabla_{e_j}^E s_2\right) \\
&= \sum_{j=1}^{q} \left(c\left(e_j\right) \nabla_{e_j}^E s_1, s_2\right) + \left(c\left(e_j\right) s_1, \nabla_{e_j}^E s_2\right) \\
&= \sum_{j=1}^{q} \left(\nabla_{e_j}^E \left(c\left(e_j\right) s_1\right), s_2\right) - \left(c\left(\nabla_{e_j}^\perp e_j\right) s_1, s_2\right) + \left(c\left(e_j\right) s_1, \nabla_{e_j}^E s_2\right) \\
&= \left(\sum_{j=1}^{q} \nabla_{e_j}^\perp \left(c\left(e_j\right) s_1, s_2\right)\right) - \left(c\left(\sum_{j=1}^{q} \nabla_{e_j}^\perp e_j\right) s_1, s_2\right) \\
&= -\sum_{j=1}^{q} \nabla_{e_j}^\perp i_{e_j} \omega + \omega\left(\sum_{j=1}^{q} \nabla_{e_j}^\perp e_j\right),
\end{aligned}$$

where ω is the basic form defined by $\omega\left(X\right) = -\left(c\left(X\right) s_1, s_2\right)$ for $X \in C^\infty\left(Q\right)$. Continuing,

$$\begin{aligned}
\left(D_{\mathrm{tr}}^E s_1, s_2\right) - \left(s_1, D_{\mathrm{tr}}^E s_2\right) &= -\sum_{j=1}^{q} \nabla_{e_j}^\perp i_{e_j} \omega + \omega\left(\sum_{j=1}^{q} \nabla_{e_j}^\perp e_j\right) \\
&= -\sum_{j=1}^{q} \left(i_{e_j} \nabla_{e_j}^\perp + i_{\nabla_{e_j}^\perp e_j}\right) \omega + \omega\left(\sum_{j=1}^{q} \nabla_{e_j}^\perp e_j\right) \\
&= -\sum_{j=1}^{q} i_{e_j} \nabla_{e_j}^\perp \omega.
\end{aligned}$$

Note we have been using the normal Levi-Civita connection $\nabla^\perp$. If we (locally) complete the normal frame field to an orthonormal frame field $\{e_1, ..., e_n\}$ for TM near $x \in M$. Letting $\nabla^M = \nabla^\perp + \nabla^{\tan}$ be the Levi-Civita connection on $\Omega\left(M\right)$, the divergence of a general basic one-form β is

$$\begin{aligned}
\delta\beta &= -\sum_{j=1}^{n} i_{e_j} \nabla_{e_j}^M \beta \\
&= -\sum_{j=1}^{n} i_{e_j} \nabla_{e_j}^\perp \beta + -\sum_{j=1}^{n} i_{e_j} \nabla_{e_j}^{\tan} \beta \\
&= -\sum_{j=1}^{q} i_{e_j} \nabla_{e_j}^\perp \beta + -\sum_{j>q}^{n} i_{e_j} \nabla_{e_j}^{\tan} \beta.
\end{aligned}$$

Letting $\beta=\sum_{k=1}^{q}\beta_k e_k^*$, then each β_k is basic and

$$\begin{aligned}
\delta\beta &= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta-\sum_{j>q}^{n} i_{e_j}\nabla_{e_j}^{\tan}\left(\sum_{k=1}^{q}\beta_k e_k^*\right)\\
&= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta-\sum_{k=1}^{q}\sum_{j>q}^{n}\beta_k i_{e_j}\nabla_{e_j}^{\tan}\left(e_k^*\right)\\
&= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta-\sum_{k=1}^{q}\sum_{j>q}\beta_k i_{e_j}\left(\sum_{m>q}\left(\nabla_{e_j}^{M}\left(e_k^*\right),e_m^*\right)e_m^*\right)\\
&= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta+\sum_{k=1}^{q}\sum_{j>q}\beta_k i_{e_j}\left(\sum_{m>q}\left(\nabla_{e_j}^{M}\left(e_m^*\right),e_k^*\right)e_m^*\right)\\
&= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta+\sum_{k=1}^{q}\sum_{j>q}\beta_k\left(\sum_{m>q}\left(\nabla_{e_j}^{M}\left(e_j^*\right),e_k^*\right)\right)\\
&= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta+i_H\beta,
\end{aligned}$$

where H is the mean curvature vector field of the foliation. Thus, for every basic one-form β,

$$-\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\beta=\delta\beta-i_H\beta.$$

Applying this result to the form ω defined above, we have

$$\begin{aligned}
\left(D_{\mathrm{tr}}^{E}s_1,s_2\right)-\left(s_1,D_{\mathrm{tr}}^{E}s_2\right) &= -\sum_{j=1}^{q} i_{e_j}\nabla_{e_j}^{\perp}\omega\\
&= \delta\omega-i_H\omega\\
&= \delta\omega+\left(c\left(H\right)s_1,s_2\right)\\
&= \delta\omega-\left(s_1,c\left(H\right)s_2\right).
\end{aligned}$$

Next, letting $P\colon L^2\left(\Omega\left(M\right)\right)\to L^2\left(\Omega_b\left(M,\mathcal{F}\right)\right)$ denote the orthogonal projection onto the closure of basic forms in $L^2\left(\Omega\left(M\right)\right)$, we observe that $\delta_b=P\delta$ is the adjoint of d_b, the restriction of the exterior derivative to basic forms. Using the results of [49], P maps smooth forms to smooth basic forms, and the projection of the smooth function $\left(s_1,c\left(H\right)s_2\right)$ is simply $\left(s_1,c\left(H_b\right)s_2\right)$, where H_b is the vector field $P\left(H^{\flat}\right)^{\sharp}$, the basic projection of the mean curvature vector field. If we had originally chosen our bundle-like metric to have basic mean curvature, which is always possible by [18], then $H_b=H$. In any case, the right hand side of the formula above is a basic function, so that

$$\left(D_{\mathrm{tr}}^{E}s_1,s_2\right)-\left(s_1,D_{\mathrm{tr}}^{E}s_2\right)=\delta_b\omega-\left(s_1,c\left(H_b\right)s_2\right).$$

We conclude:

Proposition 2.7. *The formal adjoint of the transversal Dirac operator is* $\left(D_{\mathrm{tr}}^{E}\right)^*=D_{\mathrm{tr}}^{E}-c\left(H_b\right)$.

Definition 2.8. *The* ***basic Dirac operator*** *associated to a basic Clifford module* $\left(E,(\cdot,\cdot),\nabla^E\right)$ *over a Riemannian foliation* $(M,\mathcal{F})$ *with bundle-like metric is*

$$D_b^E = D_{\mathrm{tr}}^E - \frac{1}{2}c\left(H_b\right): C_b^\infty\left(E\right) \to C_b^\infty\left(E\right).$$

Remark 2.9. *Note that the formal adjoint of* D_b^E *is* $\left(D_{\mathrm{tr}}^E\right)^* + \frac{1}{2}c\left(H_b\right) = D_{\mathrm{tr}}^E - \frac{1}{2}c\left(H_b\right) = D_b^E$. *Thus,* D_b^E *is formally sel-adjoint. In* [28], *the researchers showed that the eigenvalues of* D_b^E *are independent of the choice of the bundle-like metric that restricts to the given transverse metric of the Riemannian foliation.*

Let $\left(E,(\cdot,\cdot),\nabla^E\right)$ be a basic Clifford bundle over the Riemannian foliation $(M,\mathcal{F})$, and let $D_b^E\colon C_b^\infty\left(E\right) \to C_b^\infty\left(E\right)$ be the associated basic Dirac operator. Assume that $E = E^+ \oplus E^-$, with $D_b^{E\pm}\colon C_b^\infty\left(E^\pm\right) \to C_b^\infty\left(E^\mp\right)$. Let $\left(D_b^{E\pm}\right)^*: L_b^2\left(E^\mp\right) \to L_b^2\left(E^\pm\right)$ denote the adjoint of $D_b^{E\pm}$.

Definition 2.10. *The* ***analytic basic index*** *of* D_b^E *is*

$$\mathrm{ind}_b\left(D_b^E\right) = \dim\ker D_b^{E+}\big|_{L^2\left(C_b^\infty(E^+)\right)} - \dim\ker\left(D_b^{E-}\right)^*\big|_{L^2\left(C_b^\infty(E^-)\right)}.$$

Remark 2.11. *At this point, it is not clear that these dimensions are finite. We demonstrate this fact inside this section.*

2.5. **Examples.** The standard examples of ordinary Dirac operators are the spin^c Dirac operator, the de Rham operator, the signature operator, and the Dolbeault operator. Transversally elliptic analogues of these operators and their corresponding basic indices are typical examples of basic Dirac operators.

Suppose that the normal bundle $Q = TM\diagup T\mathcal{F} \to M$ of the Riemannian foliation $(M,\mathcal{F})$ is spin^c. Then there exists a foliated Hermitian basic Clifford bundle $\left(S,(\cdot,\cdot),\nabla^S\right)$ over M such that for all $x\in M$, S_x is isomorphic to the standard spinor representation of the Clifford algebra $\mathbb{C}\mathrm{l}\left(Q_x\right)$ (see [42]). The associated basic Dirac operator $\not{\partial}_b^S$ is called a **basic spin^c Dirac operator**. The meaning of the integer $\mathrm{ind}_b\left(\not{\partial}_b^S\right)$ is not clear, but it is an obstruction to some transverse curvature and other geometric conditions (see [26], [30], [41], [28]).

Suppose $\mathcal{F}$ has codimension q. The **basic Euler characteristic** is defined as

$$\chi\left(M,\mathcal{F}\right) = \sum_{k=0}^{q}(-1)^k \dim H^k\left(M,\mathcal{F}\right),$$

provided that all of the basic cohomology groups $H^k\left(M,\mathcal{F}\right)$ are finite-dimensional. Although $H^0\left(M,\mathcal{F}\right)$ and $H^1\left(M,\mathcal{F}\right)$ are always finite-dimensional, there are foliations for which higher basic cohomology groups can be infinite-dimensional. For example, in [25], the author gives an example of a flow on a 3-manifold for which $H^2\left(M,\mathcal{F}\right)$ is infinite-dimensional. There are various proofs that the basic cohomology of a Riemannian foliation on a closed manifold is finite-dimensional; see for example [22] for the original proof using spectral sequence techniques or [37] and [49] for proofs using a basic version of the Hodge theorem.

It is possible to express the basic Euler characteristic as the index of an operator. Let d_b denote the restriction of the exterior derivative d to basic forms over the Riemannian foliation $(M,\mathcal{F})$ with bundle-like metric, and let δ_b be the adjoint of d_b. It can be shown that δ_b is the restriction of the operator $P\delta$ to basic forms, where δ is the adjoint of d on all forms and P is the L^2 -orthogonal projection of the space of forms onto the space of basic forms. For general foliations, this is not a smooth operator, but in the case of Riemannian foliations, P maps smooth forms to smooth basic forms (see [49]), and $P\delta$ is a differential operator. In perfect analogy to the fact that the index of the de Rham operator

$$d+\delta\colon \Omega^{\mathrm{even}}(M)\to\Omega^{\mathrm{odd}}(M)$$

is the ordinary Euler characteristic, it can be shown that the basic index of the differential operator $d+P\delta$, that is the index of

$$D_b' = d_b+\delta_b\colon \Omega_b^{\mathrm{even}}(M,\mathcal{F})\to\Omega_b^{\mathrm{odd}}(M,\mathcal{F}),$$

is the basic Euler characteristic. The same proof works; this time we must use the basic version of the Hodge theorem (see [22], [37], and [49]). Note that the equality of the basic index remains valid for nonRiemannian foliations; however, the Fredholm property fails in many circumstances. It is interesting to note that the operator $d_b+\delta_b$ fails to be transversally elliptic in some examples of nonRiemannian foliations.

The principal symbol of D_b' is as follows. We define the Clifford multiplication of $\mathbb{C}\mathrm{l}(Q)$ on the bundle $\wedge^*Q^*$ by the action

$$v\cdot\ = \left(v^\flat\wedge\right) - \left(v\lrcorner\right)$$

for any vector $v\in N\mathcal{F}\cong Q$. With the standard connection and inner product defined by the metric on Q, the bundle $\wedge^*Q^*$ is a basic Clifford bundle. The corresponding basic Dirac operator, called the **basic de Rham operator** on basic forms, satisfies

$$D_b = d+\delta_b-\frac{1}{2}\left(\kappa_b\wedge+\kappa_b\lrcorner\right) = D_b'-\frac{1}{2}\left(\kappa_b\wedge+\kappa_b\lrcorner\right).$$

The kernel of this operator represents the twisted basic cohomology classes, the cohomology of basic forms induced by the differential $\widetilde{d}$ defined as

$$\widetilde{d} = d-\frac{1}{2}\kappa_b\wedge .$$

See [29] for an extended discussion of twisted basic cohomology, the basic de Rham operator, and its properties. We have $\mathrm{ind}_b(D_b') = \mathrm{ind}_b(D_b)$ because they differ by a zeroth order operator (see the Fredholm properties of the basic index in Section 3.1 below), and thus

$$\mathrm{ind}_b(D_b') = \mathrm{ind}_b(D_b) = \chi(M,\mathcal{F}),$$

the basic Euler characteristic of the complex of basic forms.

3. Fredholm properties and equivariant theory

3.1. **Molino theory and properties of the basic index.** Let $\widehat{M}\xrightarrow{p}M$ denote the principal bundle of ordered pairs of frames (ϕ_x,ψ_x) over $x\in M$, where $\phi_x\colon\mathbb{R}^q\to N_x\mathcal{F}$ is an isometry and $\psi\colon\mathbb{C}^k\to E_x$ is a complex isometry. This is a principal G–bundle, where $G\cong O(q)\times U(k)$, and it comes equipped with a natural metric connection ∇ associated to the Riemannian and Hermitian structures of $E\to M$. The foliated vector bundles $Q\to(M,\mathcal{F})$ and $E\to(M,\mathcal{F})$ naturally give $\widehat{M}$ the structure of a foliated principal bundle with lifted foliation $\widehat{\mathcal{F}}$. Transferring the normalized, biinvariant metric on G to the fibers and using the connection ∇, we define a natural metric $(\cdot,\cdot)_{\widehat{M}}$ on $\widehat{M}$ that is locally a product. The connection ∇^E pulls back to a basic connection ∇^{p^*E} on p^*E; the horizontal subbundle $\mathcal{H}p^*E$ of Tp^*E is the inverse image of the horizontal subbundle $\mathcal{H}E\subset TE$ under the natural map $Tp^*E\to TE$. It is clear that the metric is bundle-like for the lifted foliation $\widehat{\mathcal{F}}$.

Observe that the foliation $\widehat{\mathcal{F}}$ is transversally parallelizable, meaning that the normal bundle of the lifted foliation is parallelizable by $\widehat{\mathcal{F}}$-basic vector fields. To see this, we use a modification of the standard construction of the parallelism of the frame bundle of a manifold (see [48, p.82] for this construction in the case where the principle bundle is the bundle of transverse orthonormal frames). Let $G = O(q) \times U(k)$, and let θ denote the $\mathbb{R}^q$–valued solder form of $\widehat{M} \to M$. Given the pair of frames $z = (\phi, \psi)$ where $\phi \colon \mathbb{R}^q \to N_{p(z)}\mathcal{F}$ and $\psi : \mathbb{C}^k \to E_{p(z)}$ and given $X_z \in T_z\widehat{M}$, we define $\theta(X_z) = \phi^{-1}\left(\pi^{\perp} p_* X_z\right)$, where $\pi^{\perp} \colon T_{p(z)}M \to N_{p(z)}\mathcal{F}$ is the orthogonal projection. Let ω denote the $\mathfrak{o}(q) \oplus \mathfrak{u}(k)$–valued connection one-form. Let $\{e_1, \ldots, e_q\}$ be the standard orthonormal basis of $\mathbb{R}^q$, and let $\{E_j\}_{j=1}^{\dim G}$ denote a fixed orthonormal basis of $\mathfrak{o}(q) \oplus \mathfrak{u}(k)$. We uniquely define the vector fields $V_1, ..., V_q, \overline{E_1}, ..., \overline{E_{\dim G}}$ on $\widehat{M}$ by the conditions

(1) $V_i \in N_z\widehat{\mathcal{F}}$, $\overline{E_j} \in N_z\widehat{\mathcal{F}}$ for every i, j.
(2) $\omega(V_i) = 0$, $\omega\left(\overline{E_j}\right) = E_j$ for every i, j.
(3) $\theta(V_i) = e_i$, $\theta\left(\overline{E_j}\right) = 0$ for every i, j.

Then the set of $\widehat{\mathcal{F}}$–basic vector fields $\left\{V_1, ..., V_q, \overline{E_1}, ..., \overline{E_{\dim G}}\right\}$ is a transverse parallelism on $\left(\widehat{M}, \widehat{\mathcal{F}}\right)$ associated to the connection ∇. By the fact that $\left(\widehat{M}, \widehat{\mathcal{F}}\right)$ is Riemannian and the structure theorem of Molino [48, Chapter 4], the leaf closures of $\left(\widehat{M}, \widehat{\mathcal{F}}\right)$ are the fibers of a Riemannian submersion $\widehat{\pi} \colon \widehat{M} \to \widehat{W}$.

Next, we show that the bundle $p^*E \to \widehat{M}$ is $G_{\widehat{\mathcal{F}}}$–equivariant. An element of the foliation groupoid $G_{\widehat{\mathcal{F}}}$ is a triple of the form $(y, z, [\cdot])$, where y and z are points of a leaf of $\widehat{\mathcal{F}}$ and $[\cdot]$ is the set of all piecewise smooth curves starting at y and ending at z, since all such curves are equivalent because the holonomy is trivial on $\widehat{M}$. The basic connection on E induces a $G_{\widehat{\mathcal{F}}}$–action on p^*E, defined as follows. Given a vector $(y, v) \in (p^*E)_y$ so that $y \in \widehat{M}$, $v \in E_{p(y)}$, we define the action of $\widehat{g} = (y, z, [\cdot])$ by $S_{\widehat{g}}(y, v) = (z, P_\gamma v)$, where γ is any piecewise smooth curve from $p(y)$ to $p(z)$ in the leaf containing $p(y)$ that lifts to a leafwise curve in $\widehat{M}$ from y to z and where P_γ denotes parallel translation in E along the curve γ. It is easy to check that this action makes p^*E into a $G_{\widehat{\mathcal{F}}}$–equivariant, foliated vector bundle. The pullback p^* maps basic sections of E to basic sections of p^*E. Also, the $O(q) \times U(k)$–action on $\left(\widehat{M}, \ \widehat{\mathcal{F}}\right)$ induces an action of $O(q) \times U(k)$ on p^*E that preserves the basic sections.

Observe that if $s \in C_b^\infty(E)$, then p^*s is a basic section of p^*E that is $O(q) \times U(k)$–invariant. Conversely, if $\widehat{s} \in C^\infty(p^*E)$ is $O(q) \times U(k)$–invariant, then $\widehat{s} = p^*s$ for some $s \in C^\infty(E)$. Next, suppose $\widehat{s} = p^*s$ is $O(q) \times U(k)$–invariant and basic. Given any vector $X \in T_{p(y)}\mathcal{F}$ and its horizontal lift $\widetilde{X} \in T_y\widehat{\mathcal{F}}$, we have

$$0 = \nabla^{p^*E}_{\widetilde{X}}\widehat{s} = \nabla^{p^*E}_{\widetilde{X}} p^*s = p^*\nabla^E_X s,$$

so that s is also basic. We have shown that $C_b^\infty(E)$ is isomorphic to $C_b^\infty\left(\widehat{M}, p^*E\right)^{O(q)\times U(k)}$.

We now construct a Hermitian vector bundle $\mathcal{E}$ over $\widehat{W}$, similar to the constructions in [52] and [21]. Given $w \in \widehat{W}$ and the corresponding leaf closure $\widehat{\pi}^{-1}(w) \in \widehat{M}$, consider a basic section $s \in C_b^\infty\left(\widehat{M}, p^*E\right)$ restricted to $\widehat{\pi}^{-1}(w)$. Given any $y \in \widehat{M}$, the vector $s(y)$ uniquely determines s on the entire leaf closure by parallel transport, because the section is smooth.

Similarly, given a vector $v_y \in (p^*E)_y$, there exists a basic section $s \in C_b^\infty\left(\widehat{M}, p^*E\right)$ such that $s(y) = v_y$, because there is no obstruction to extending, by the following argument. Given a basis $\{b_1, ..., b_k\}$ of $\mathbb{C}^k$, we define the k linearly independent, basic sections s_k of p^*E by $s_j((\phi,\psi)) = \psi(b_j) \in (p^*E)_{(\phi,\psi)} = E_{p((\phi,\psi))}$. Thus, given a local frame $\{v_j\}$ for p^*E on a $\widehat{\mathcal{F}}$ –transversal submanifold near y, there is a unique extension of this frame to be a frame consisting of basic sections on a tubular neighborhood of the leaf closure containing y; in particular a vector may be extended to be a basic section of p^*E. We now define $\mathcal{E}_w = C_b^\infty\left(\widehat{M}, p^*E\right) \diagup \sim_w$, where two basic sections s, $s'\colon \widehat{M} \to p^*E$ are equivalent($s \sim_w s'$) if $s(y) = s'(y)$ for every $y \in \widehat{\pi}^{-1}(w)$. By the reasoning above, $\mathcal{E}_w$ is a complex vector space whose dimension is equal to the complex rank of $p^*E \to \widehat{M}$. Alternately, we could define $\mathcal{E}_w$ to be the vector space of $\widehat{\mathcal{F}}$ -basic sections of p^*E restricted to the leaf closure $\widehat{\pi}^{-1}(w)$. The union $\cup_{w\in\widehat{W}} \mathcal{E}_w$ forms a smooth, complex vector bundle $\mathcal{E}$ over $\widehat{W}$; local trivializations of $\mathcal{E}$ are given by local, basic framings of the trivial bundle $p^*E \to \widehat{M}$. We remark that in the constructions of [52] and [21], the vector bundle was lifted to the transverse orthonormal frame bundle $\widehat{M}$, and in that case the corresponding bundle $\mathcal{E}$ in those papers could have smaller rank than E.

We let the invertible $\Phi\colon C^\infty\left(\widehat{W}, \mathcal{E}\right) \to C_b^\infty\left(\widehat{M}, p^*E, \widehat{\mathcal{F}}\right)$ be the almost tautological map defined as follows. Given a section $\widehat{s}$ of $\mathcal{E}$, its value at each $w \in \widehat{W}$ is an equivalence class $[s]_w$ of basic sections. We define for each $y \in \widehat{\pi}^{-1}(w)$,

$$\Phi(\widehat{s})(y) = s(y) \in (p^*E)_y .$$

By the continuity of the basic section s, the above is independent of the choice of this basic section in the equivalence class. By the definition of Φ and of the trivializations of $\mathcal{E}$, it is clear that Φ is a smooth map. Also, the $G = O(q) \times U(k)$ action on basic sections of p^*E pushes forward to a G action on sections of $\mathcal{E}$. We have the following commutative diagram, with $W = M\diagup\overline{\mathcal{F}} = \widehat{W}\diagup G$ the leaf closure space of $(M, \mathcal{F})$.

$$\begin{array}{ccccc}
p^*E & & & & \mathcal{E} \\
& \searrow & & & \downarrow \\
G & \hookrightarrow & \left(\widehat{M}, \widehat{\mathcal{F}}\right) & \xrightarrow{\widehat{\pi}} & \widehat{W} \\
& & \downarrow p & \circlearrowleft & \downarrow \\
E & \to & (M, \mathcal{F}) & \longrightarrow & W .
\end{array}$$

Observe that we have the necessary data to construct the basic Dirac operator on sections of p^*E over $\widehat{M}$ corresponding to the pullback foliation $p^*\mathcal{F}$ on $\widehat{M}$. The connection ∇^{p^*E} is a basic connection with respect to this Riemannian foliation, and the normal bundle $N(p^*\mathcal{F})$ projects to the normal bundle $Q = N\mathcal{F}$, so that the action of $\mathbb{C}\mathrm{l}(Q)$ on E lifts to an action of $\mathbb{C}\mathrm{l}(N(p^*\mathcal{F}))$ on p^*E. Using this basic Clifford bundle structure, we construct the transversal Dirac-type operator $D_{\mathrm{tr},p^*}^{p^*E}$ and the basic Dirac-type operator $D_{b,p^*}^{p^*E}$ on $C_b^\infty\left(\widehat{M}, p^*E, p^*\mathcal{F}\right)$; we add the subscript p^* to emphasize that we are working with $p^*\mathcal{F}$ rather than the lifted foliation. Observe that $C_b^\infty\left(\widehat{M}, p^*E, p^*\mathcal{F}\right) = C_b^\infty\left(\widehat{M}, p^*E, \widehat{\mathcal{F}}\right)^G \subset C_b^\infty\left(\widehat{M}, p^*E, \widehat{\mathcal{F}}\right)$. It is clear from the construction that p^* is an isomorphism from $C_b^\infty(M,E)$ to $C_b^\infty\left(\widehat{M},p^*E,p^*\mathcal{F}\right)$ and $p^*\circ D_b^E = D_{b,p^*}^{p^*E}\circ p^*$.

We define the operator $\mathcal{D}\colon C^\infty\left(\widehat{W},\mathcal{E}\right)\to C^\infty\left(\widehat{W},\mathcal{E}\right)$ by

$$\mathcal{D}=\Phi^{-1}\circ\left(D_{\mathrm{tr},p^*}^{p^*E}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi,$$

where $\widehat{H_b}$ is the basic mean curvature of the pullback foliation, which is merely the horizontal lift of H_b. Let $\mathcal{D}^G$ denote the restriction of $\mathcal{D}$ to $C^\infty\left(\widehat{W},\mathcal{E}\right)^G$. Note that

$$\Phi\colon C^\infty\left(\widehat{W},\mathcal{E}\right)^G\to C_b^\infty\left(\widehat{M},p^*E,\widehat{\mathcal{F}}\right)^G$$

is an isomorphism. Observe that the Hermitian metric on p^*E induces a well-defined Hermitian metric on $\mathcal{E}$ that is invariant under the action of G.

Assume that $E=E^+\oplus E^-$ with $D_b^{E\pm}\colon C_b^\infty(M,E^\pm)\to C_b^\infty(M,E^\mp)$. We define $D_{b,p^*}^{\pm}$ to be the restrictions

$$\left(D_{\mathrm{tr},p^*}^{p^*E}-\frac{1}{2}c\left(\widehat{H_b}\right)\right):C_b^\infty\left(\widehat{M},p^*E^\pm,p^*\mathcal{F}\right)\to C_b^\infty\left(\widehat{M},p^*E^\mp,p^*\mathcal{F}\right),$$

We define the bundles $\mathcal{E}^\pm$ and the operator

$$\mathcal{D}^+=\Phi^{-1}\circ\left(D_{\mathrm{tr},p^*}^{p^*E}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi\colon C^\infty\left(\widehat{W},\mathcal{E}^+\right)\to C^\infty\left(\widehat{W},\mathcal{E}^-\right)\tag{3.1}$$

in an analogous way. We now have the following result.

Proposition 3.1. *Let $D_b^{E+}\colon C_b^\infty(M,E^+)\to C_b^\infty(M,E^-)$ be a basic Dirac operator for the rank k complex vector bundle $E=E^+\oplus E^-$ over the transversally oriented Riemannian foliation $(M,\mathcal{F})$, and let $G=O(q)\times U(k)$. Then*

$$\mathrm{ind}_b\left(D_b^{E+}\right)=\mathrm{ind}\left(\mathcal{D}^G\right),$$

where $\mathrm{ind}\left(\mathcal{D}^G\right)$ *refers to the index of the transversally elliptic operator $\mathcal{D}^+$ restricted to G–invariant sections (equivalently, the supertrace of the invariant part of the virtual representation–valued equivariant index of the operator $\mathcal{D}$). It is not necessarily the case that the adjoint $\mathcal{D}^-\colon C^\infty\left(\widehat{W},\mathcal{E}^-\right)\to C^\infty\left(\widehat{W},\mathcal{E}^+\right)$ coincides with $\Phi^{-1}\circ\left(D_{\mathrm{tr},p^*}^{p^*E}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi\Big|_{C^\infty\left(\widehat{W},\mathcal{E}^-\right)}$, but the principal transverse symbols of $\mathcal{D}^+$ and $\mathcal{D}^-$ evaluated on a normal space to an orbit in $\widehat{W}$ correspond with the restriction of the principal transverse symbol of D_b^{E+} and D_b^{E-} restricted to the normal space to a leaf closure in M.*

Proof. The kernels satisfy

$$\begin{aligned}\ker(D_b^{E+})&\cong\ker\left(p^*\circ D_b^{E+}\right)\\&\cong\ker\left(D_{b,p^*}^{+}\circ p^*\big|_{C_b^\infty(M,E^+)}\right)\\&\cong\ker\left(\left(D_{\mathrm{tr},p^*}^{p^*E}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ p^*\Big|_{C_b^\infty(M,E^+)}\right)\end{aligned}$$

$$\cong \ker\left(\Phi^{-1}\circ\left(D^{p^*E}_{\mathrm{tr},p^*}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi\Big|_{C^\infty\left(\widehat{W},\mathcal{E}^+\right)^G}\right)$$
$$\cong \ker\left(\mathcal{D}^{+G}\right),$$

the kernel of the operator restricted to G –invariant sections. Next, while D_b^{E-} is the adjoint of D_b^{E+} with respect to the L^2-inner product on the closure of the space of basic sections of E, it is not necessarily true that the adjoint of $\mathcal{D}^+=\Phi^{-1}\circ\left(D^{p^*E}_{\mathrm{tr},p^*}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi\Big|_{C^\infty\left(\widehat{W},\mathcal{E}^+\right)}$ is $\Phi^{-1}\circ\left(D^{p^*E}_{\mathrm{tr},p^*}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi\Big|_{C^\infty\left(\widehat{W},\mathcal{E}^-\right)}$, because although the operators have the same principal transverse symbol, the volumes of the orbits on $\widehat{W}$ need not coincide with the volumes of the leaf closures on M, at least with the metric on $\widehat{W}$ that we have chosen. However, it is possible to choose a different metric, similar to that used in [52, Theorem 3.3], so by using the induced L^2-metric on invariant sections of $\mathcal{E}$ over $\widehat{W}$ and the L^2 metric on basic sections of E on M, Φ is an isometry. Specifically, let $\phi\colon \widehat{W}\to\mathbb{R}$ be the smooth positive function defined by $\phi(w)=\mathrm{vol}\left(\widehat{\pi}^{-1}(w)\right)$. Let $d_{\widehat{W}}$ be the dimension of $\widehat{W}$. We determine a new metric g' on $\widehat{W}$ by conformally multiplying the original metric g on $\widehat{W}$ by $\phi^{2/d_{\widehat{W}}}\in C^\infty\left(\widehat{W}\right)$, so that the volume form on $\widehat{W}$ is multiplied by ϕ. Note that $\phi(w)\,\mathrm{vol}_g(\mathcal{O}_w)=\mathrm{vol}\left(\overline{L}\right)$ by the original construction, where $\overline{L}=p\left(\widehat{\pi}^{-1}(w)\right)$ is the leaf closure corresponding to the orbit $\mathcal{O}_w=wG\subset\widehat{W}$. By using the new metric g' on $\widehat{W}$, we see that Φ extends to an L^2-isometry and that G still acts by isometries on $\widehat{W}$. Then

$$\ker\left(D_b^{E-}\right)\cong\ker\left(\Phi^{-1}\circ\left(D^{p^*E}_{\mathrm{tr},p^*}-\frac{1}{2}c\left(\widehat{H_b}\right)\right)\circ\Phi\Big|_{C^\infty\left(\widehat{W},\mathcal{E}^-\right)^G}\right)$$
$$\cong\ker\left(\mathcal{D}^{G,\mathrm{adj}'}\right),$$

where the superscript adj' refers to the adjoint with respect to the L^2 metrics $C^\infty\left(\widehat{W},\mathcal{E}^\pm\right)^G$ induced by g'. Therefore, the analytic basic index satisfies

$$\mathrm{ind}_b\left(D_b^{E+}\right)=\mathrm{ind}'\left(\mathcal{D}^G\right),$$

where $\mathrm{ind}'\left(\mathcal{D}^G\right)$ is the analytic index of the transversally elliptic operator $\mathcal{D}$ restricted to G-invariant sections, with adjoint calculated with respect to the choice of metric g'. Because the restriction of $\mathcal{D}$ to G-invariant sections is a Fredholm operator (see [1]), $\mathrm{ind}'\left(\mathcal{D}^G\right)$ is independent of the choice of metric. □

The Fredholm properties of the equivariant index of transversally elliptic operators (see [1]) imply the following.

Corollary 3.2. *In the notation of Proposition 3.1, the analytic basic index* $\mathrm{ind}_b\left(D_b^{E+}\right)$ *is a well-defined integer. Further, it is invariant under smooth deformations of the basic operator and metrics that preserve the invertibility of the principal symbol* $\sigma(\xi_x)$ *of* D_b^{E+} *for every* $x\in M$*, but only for* $\xi_x\in\overline{Q}_x^*=\left(T_xM\diagup T_x\overline{L}_x\right)^*$*, the dual to the normal space to the leaf closure through* x*.*

Note that if f is a smooth function on $\widehat{W}$ such that df_w is an element of the dual space to the normal bundle to the orbit space at $w \in \widehat{W}$, and if $\widehat{s}$ is a smooth section of $\mathcal{E}^+$, then

$$\begin{aligned}\left[\mathcal{D}^+, f\right] \widehat{s} &= \Phi^{-1} c \left(d \left(\widehat{\pi}^* f \right)^{\#} \right) \Phi \left(\widehat{s} \right) \\ &= \Phi^{-1} c \left(\left(\widehat{\pi}^* df \right)^{\#} \right) \Phi \left(\widehat{s} \right) \\ &= \widehat{c} \left(df^{\#} \right) \widehat{s}.\end{aligned}$$

This implies that $\mathcal{D}^+$ is a Dirac operator on sections of $\mathcal{E}^+$, since $\left(\mathcal{D}^+\right)^* \mathcal{D}^+$ is a generalized Laplacian. The analogous result is true for $\mathcal{D}^-$.

It is possible use the Atiyah–Segal Theorem ([4]) to compute $\operatorname{ind}^G \left(\mathcal{D}^+\right)$, but only in the case where $\mathcal{D}$ is a genuinely elliptic operator. Recall that if D' is an elliptic operator on a compact, connected manifold M that is equivariant with respect to the action of a compact Lie group G', then G' represents on both finite–dimensional vector spaces $\ker D'$ and $\ker D'^*$ in a natural way. For $g \in G'$,

$$\operatorname{ind}_g \left(D'\right) := \operatorname{tr}\left(g \middle| \ker D'\right) - \operatorname{tr}\left(g \middle| \ker D'^*\right).$$

where dg is the normalized, bi-invariant measure on G'. The Atiyah-Segal Theorem computes this index in terms of an integral over the fixed point set of g. We will use our equivariant index theorem in [14] to evaluate $\operatorname{ind}^G \left(\mathcal{D}^+\right)$ in terms of geometric invariants of the operator restricted to the strata of the foliation.

Remark 3.3. *Under the additional assumption that $E \to (M, \mathcal{F})$ is $G_{\mathcal{F}}$–equivariant, the pullback of E to the transverse orthonormal frame bundle is already transversally parallelizable. Thus, it is unnecessary in this case to pull back again to the unitary frame bundle. We may then replace G by $O(q)$, and $\widehat{W}$ is the so-called* ***basic manifold*** *of the foliation, as in the standard construction in* [48].

Remark 3.4. *If $(M, \mathcal{F})$ is in fact transversally orientable, we may replace $O(q)$ with $SO(q)$ and work with the bundle of oriented orthonormal frames.*

3.2. **The asymptotic expansion of the trace of the basic heat kernel .** In this section, we will state some results concerning the spectrum of the square of a basic Dirac-type operator and the heat kernel corresponding to this operator, which are corollaries of the work in the previous section and are of independent interest.

Proposition 3.5. *Let $D_b^{E+} \colon C_b^\infty \left(M, E^+\right) \to C_b^\infty \left(M, E^-\right)$ be a basic Dirac operator for the rank k complex vector bundle $E = E^+ \oplus E^-$ over the transversally oriented Riemannian foliation $(M, \mathcal{F})$, and let $\left(D_b^{E+}\right)^{\mathrm{adj}}$ be the adjoint operator. Then the operators*

$$\begin{aligned}L^+ &= \left(D_b^{E+}\right)^{\mathrm{adj}} D_b^{E+}, \\ L^- &= D_b^{E+} \left(D_b^{E+}\right)^{\mathrm{adj}}\end{aligned}$$

are essentially self–adjoint, and their spectrum consists of nonnegative real eigenvalues with finite multiplicities. Further, the operators $L^\pm$ have the same positive spectrum, including multiplicities.

Proof. By (3.1) and the proof of Proposition 3.1, the operators L^+ and L^- are conjugate to essentially self-adjoint, second order, G-equivariant, transversally elliptic operators on $\widehat{W}$. □

The basic heat kernel $K_b(t,x,y)$ for L is a continuous section of $E \boxtimes E^*$ over $\mathbb{R}_{>0} \times M \times M$ that is C^1 with respect to t, C^2 with respect to x and y, and satisfies, for any vector $e_y \in E_y$,

$$\left(\frac{\partial}{\partial t} + L_x\right) K_b(t,x,y)e_y = 0$$

$$\lim_{t \to 0^+} \int_M K_b(t,x,y)\, s(y)\, dV(y) = s(x)$$

for every continuous basic section $s\colon M \to E$. The principal transverse symbol of L satisfies $\sigma\left(L\right)\left(\xi_x\right) = \left|\xi_x\right|^2 I_x$ for every $\xi_x \in N_x\mathcal{F}$, where $I_x\colon E_x \to E_x$ is the identity operator. The existence of the basic heat kernel has already been shown in [51]. Let $\overline{q}$ be the codimension of the leaf closures of $(M,\mathcal{F})$ with maximal dimension. The following theorems are consequences of [12], [13] and the conjugacy mentioned in the proof of the proposition above.

Theorem 3.6. *Under the assumptions in Proposition 3.5, let* $0 < \lambda_0^b \leq \lambda_1^b \leq \lambda_2^b \leq ...$ *be the eigenvalues of* $L\left|_{C_b^\infty(E)}\right.$*, counting multiplicities. Then the spectral counting function* $N_b\left(\lambda\right)$ *satisfies the asymptotic formula*

$$\begin{aligned} N_b\left(\lambda\right) &:= \#\left\{ \lambda_m^b \middle| \, \lambda_m^b < \lambda \right\} \\ &\sim \frac{\operatorname{rank}\left(\mathcal{E}\right) V_{tr}}{\left(4\pi\right)^{\overline{q}/2} \Gamma\left(\frac{\overline{q}}{2}+1\right)} \lambda^{\overline{q}/2}. \end{aligned}$$

Theorem 3.7. *Under the assumptions in Proposition 3.5, the heat operators* e^{-tL^+} *and* e^{-tL^-} *are trace class, and they satisfy the following asymptotic expansions. Then, as* $t \to 0$,

$$\mathrm{Tr} e^{-tL^\pm} = K_b^\pm(t) \sim \frac{1}{t^{\overline{q}/2}} \left(a_0^\pm + \sum_{\substack{j \geq 1 \\ 0 \leq k < K_0}} a_{j,k}^\pm t^{j/2} (\log t)^k \right),$$

where K_0 *is less than or equal to the number of different dimensions of closures of infinitesimal holonomy groups of the leaves of* $\mathcal{F}$.

Remark 3.8. (The basic zeta function and determinant of the generalized basic Laplacian) *We remark that due to the singular asymptotics lemma of* [15], *we have that* $a_{jk} = 0$ *for* $j \leq \overline{q}$, $k > 0$. *We conjecture that all the logarithmic terms vanish. Note that the fact that* $a_{\overline{q}k} = 0$ *for* $k > 0$ *implies that the corresponding zeta function* $\zeta_L\left(z\right)$ *is regular at* $z = 0$, *so that the regularized determinant of* L *may be defined.*

3.3. **Stratifications of G-manifolds and Riemannian foliations.** In the following, we will describe some standard results from the theory of Lie group actions and Riemannian foliations (see [9], [38], [48]). Such G-manifolds and Riemannian foliations are stratified spaces, and the stratification can be described explicitly. In the following discussion, we often have in mind that $\widehat{W}$ is the basic manifold corresponding to $(M,\mathcal{F})$ described in the last section, but in fact the ideas apply to any Lie group G acting on a smooth, closed, connected manifold $\widehat{W}$. In the context of this paper, either G is $O\left(q\right)$, $SO\left(q\right)$, or the product of one of these with $U\left(k\right)$. We will state the results for general G and then specialize to the case of

Riemannian foliations $(M,\mathcal{F})$ and the associated basic manifold. We also emphasize that our stratification of the foliation may be finer than that described in [48], because in addition we consider the action of the holonomy on the relevant vector bundle when identifying isotropy types.

Given a G-manifold $\widehat{W}$ and $w\in\widehat{W}$, an orbit $\mathcal{O}_w=\{gw : g\in G\}$ is naturally diffeomorphic to G/H_w, where $H_w=\{g\in G\,|wg=w\}$ is the (closed) isotropy or stabilizer subgroup. In the foliation case, the group H_w is isomorphic to the structure group corresponding to the principal bundle $p\colon\widehat{\pi}^{-1}(w)\to\overline{L}$, where $\overline{L}$ is the leaf closure $p\left(\widehat{\pi}^{-1}(w)\right)$ in M. Given a subgroup H of G, let $[H]$ denote the conjugacy class of H. The isotropy type of the orbit $\mathcal{O}_x$ is defined to be the conjugacy class $[H_w]$, which is well–defined independent of $w\in\mathcal{O}_x$. On any such G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G, we say that $[H]\le[K]$ if H is conjugate to a subgroup of K, and we say $[H]<[K]$ if $[H]\le[K]$ and $[H]\ne[K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0],...,[G_r]$ such that $[G_i]\le[G_j]$ implies that $i\le j$. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset $\widehat{W_0}=\widehat{W}\left([G_0]\right)$ of the manifold $\widehat{W}$, and the other orbits are called **singular**. As a consequence, every isotropy subgroup H satisfies $[G_0]\le[H]$. Let $\widehat{W_j}$ denote the set of points of $\widehat{W}$ of orbit type $[G_j]$ for each j; the set $\widehat{W_j}$ is called the **stratum** corresponding to $[G_j]$. If $[G_j]\le[G_k]$, it follows that the closure of $\widehat{W_j}$ contains the closure of $\widehat{W_k}$. A stratum $\widehat{W_j}$ is called a **minimal stratum** if there does not exist a stratum $\widehat{W_k}$ such that $[G_j]<[G_k]$ (equivalently, such that $\overline{\widehat{W_k}}\subsetneq\overline{\widehat{W_j}}$). It is known that each stratum is a G-invariant submanifold of $\widehat{W}$, and in fact a minimal stratum is a closed (but not necessarily connected) submanifold. Also, for each j, the submanifold $\widehat{W}_{\ge j}:=\bigcup\limits_{[G_k]\ge[G_j]}\widehat{W_k}$ is a closed, G-invariant submanifold.

Now, given a proper, G-invariant submanifold S of $\widehat{W}$ and $\varepsilon>0$, let $T_\varepsilon(S)$ denote the union of the images of the exponential map at s for $s\in S$ restricted to the open ball of radius ε in the normal bundle at S. It follows that $T_\varepsilon(S)$ is also G-invariant. If $\widehat{W_j}$ is a stratum and ε is sufficiently small, then all orbits in $T_\varepsilon\left(\widehat{W_j}\right)\setminus\widehat{W_j}$ are of type $[G_k]$, where $[G_k]<[G_j]$. This implies that if $j<k$, $\overline{\widehat{W_j}}\cap\overline{\widehat{W_k}}\ne\varnothing$, and $\widehat{W_k}\subsetneq\overline{\widehat{W_j}}$, then $\overline{\widehat{W_j}}$ and $\overline{\widehat{W_k}}$ intersect at right angles, and their intersection consists of more singular strata (with isotropy groups containing conjugates of both G_k and G_j).

Fix $\varepsilon>0$. We now decompose $\widehat{W}$ as a disjoint union of sets $\widehat{W}_0^\varepsilon,\dots,\widehat{W}_r^\varepsilon$. If there is only one isotropy type on $\widehat{W}$, then $r=0$, and we let $\widehat{W}_0^\varepsilon=\Sigma_0^\varepsilon=\widehat{W}_0=\widehat{W}$. Otherwise, for $j=r,r-1,...,0$, let $\varepsilon_j=2^j\varepsilon$, and let

$$\Sigma_j^\varepsilon=\widehat{W_j}\setminus\overline{\bigcup_{k>j}\widehat{W}_k^\varepsilon},\ \widehat{W}_j^\varepsilon=T_{\varepsilon_j}\left(\widehat{W_j}\right)\setminus\overline{\bigcup_{k>j}\widehat{W}_k^\varepsilon}. \tag{3.2}$$

Thus,

$$T_\varepsilon\left(\Sigma_j^\varepsilon\right)\subset\widehat{W}_j^\varepsilon,\ \Sigma_j^\varepsilon\subset\widehat{W_j}.$$

We now specialize to the foliation case. Let $(M,\mathcal{F})$ be a Riemannian foliation, and let $E\to M$ be a foliated Hermitian vector bundle over M (defined in Section 2.2). Let the G-bundle $\widehat{M}\to M$ be either the orthonormal transverse frame bundle of $(M,\mathcal{F})$ or the bundle of ordered pairs (α, β), with α a orthonormal transverse frame and β an orthonormal frame of E with respect to the Hermitian inner product on E, as in Section 3.1. In the former case, $\widehat{M} \xrightarrow{p} M$ is an $O(q)$-bundle, and in the latter case, $\widehat{M}$ is an $O(q) \times U(k)$-bundle. We also note that in the case where $(M, \mathcal{F})$ is transversally oriented, we may replace $O(q)$ with $SO(q)$ and choose oriented transverse frames. In Section 3.1, we showed that the foliation $\mathcal{F}$ lifts to a foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$, and the lifted foliation is transversally parallelizable. We chose a natural metric on $\widehat{M}$, as explained in Section 3.1. By Molino theory ([48]), the leaf closures of $\widehat{\mathcal{F}}$ are diffeomorphic and have no holonomy; they form a Riemannian fiber bundle $\widehat{M} \xrightarrow{\widehat{\pi}} \widehat{W}$ over what is called the **basic manifold** $\widehat{W}$, on which the group G acts by isometries.

We identify the spaces $\widehat{W}_i^\varepsilon$, $\widehat{W}_i^\varepsilon \big/ G$, and $\overline{\widehat{W}_i^\varepsilon} \big/ G$ with the corresponding M_i^ε, $M_i^\varepsilon \big/ \overline{\mathcal{F}}$, and $\overline{M_i^\varepsilon} \big/ \overline{\mathcal{F}}$ on M via the correspondence

$$p\left(\widehat{\pi}^{-1}\left(G\text{–orbit on } \widehat{W}\right)\right) = \text{leaf closure of } (M, \mathcal{F}).$$

The following result is contained in [48], which is a consequence of Riemannian foliation theory and the decomposition theorems of G-manifolds (see [38]). However, we note that the decomposition described below may be finer than that described in Molino, as the bundle $E \to M$ is used in our construction to construct the basic manifold, and the group acting may be larger than the orthogonal group. The action of the holonomy on the bundle may participate in the decomposition of the foliation.

Lemma 3.9. *Let $(M, \mathcal{F})$ be a Riemannian foliation with bundle-like metric. Let $\overline{\mathcal{F}}$ denote the (possibly) singular foliation by leaf closures of $\mathcal{F}$. We let $M_j = M([G_j]) = p\left(\widehat{\pi}^{-1}\left(\widehat{W}([G_j])\right)\right)$, $M_i^\varepsilon = p\left(\widehat{\pi}^{-1}\left(\widehat{W_i^\varepsilon}\right)\right)$ with $\widehat{W}$, $\widehat{W}([G_j])$, $\widehat{W_i^\varepsilon}$ defined as above on the basic manifold. Note that M_j is a stratum on M corresponding to the union of all leaf closures whose structure group of the principal bundle $p\colon \overline{\widehat{L}} \to \overline{L}$ is in $[G_j]$, where $\overline{\widehat{L}}$ is a leaf closure of $\widehat{M}$ that projects to $\overline{L}$. It follows that all the leaf closures in M_j have the same dimension. Then we have, for every $i \in \{1, \dots, r\}$ and sufficiently small $\varepsilon > 0$:*

1. $M = \coprod_{i=0}^{r} M_i^\varepsilon$ *(disjoint union).*
2. M_i^ε *is a union of leaf closures.*
3. *The manifold M_i^ε is diffeomorphic to the interior of a compact manifold with corners; the leaf closure space space $M_i^\varepsilon \big/ \overline{\mathcal{F}} \cong \widehat{W_i^\varepsilon} \big/ O(q)$ is a smooth manifold that is isometric to the interior of a triangulable, compact manifold with corners. The same is true for each Σ_i^ε, $M_i \big/ \overline{\mathcal{F}}$.*
4. *If $[G_j]$ is the isotropy type of an orbit in M_i^ε, then $j \le i$ and $[G_j] \le [G_i]$.*
5. *The distance between the submanifold M_j and M_i^ε for $j > i$ is at least ε.*

Remark 3.10. *The lemma above remains true if at each stage $T_\varepsilon(M_j)$ is replaced by any sufficiently small open neighborhood of M_j that contains $T_\varepsilon(M_j)$, that is a $\mathcal{F}$-saturated, and whose closure is a manifold with corners.*

Remark 3.11. *The additional frames of E have no effect on the stratification of M; the corresponding M_i^ε, M_i are identical whether or not the bundle $\widehat{M} \to M$ is chosen to be the $O(q)$-bundle or the $O(q) \times U(k)$-bundle. However, the isotropy subgroups and basic manifold $\widehat{W}$ are different and depend on the structure of the bundle E.*

Definition 3.12. *With notation as in this section, suppose that $[H]$ is a maximal isotropy type with respect to the partial order $\leq$. Then the closed, saturated submanifold $M([H])$ is called a* ***minimal stratum*** *of the foliation $(M,\mathcal{F})$.*

3.4. **Fine components and canonical isotropy bundles.** First we review some definitions from [14] and [32] concerning manifolds X on which a compact Lie group G acts by isometries with single orbit type $[H]$. Let X^H be the fixed point set of H, and for $\alpha \in \pi_0\left(X^H\right)$, let X_α^H denote the corresponding connected component of X^H.

Definition 3.13. *We denote $X_\alpha = GX_\alpha^H$, and X_α is called a* ***component of*** *X* ***relative to*** *G.*

Remark 3.14. *The space X_α is not necessarily connected, but it is the inverse image of a connected component of $G\diagdown X = N\diagdown X^H$ under the projection $X \to G\diagdown X$. Also, note that $X_\alpha = X_\beta$ if there exists $n \in N$ such that $nX_\alpha^H = X_\beta^H$. If X is a closed manifold, then there are a finite number of components of X relative to G.*

We now introduce a decomposition of a G-bundle $E \to X$ over a G-space with single orbit type $[H]$ that is a priori finer than the normalized isotypical decomposition. Let E_α be the restriction $E|_{X_\alpha^H}$. For $\sigma\colon H \to U(W_\sigma)$ an irreducible unitary representation, let $\sigma^n\colon H \to U(W_\sigma)$ be the irreducible representation defined by

$$\sigma^n(h) = \sigma\left(n^{-1}hn\right).$$

Let $\widetilde{N}_{[\sigma]} = \{n \in N : [\sigma^n] \text{ is equivalent to } [\sigma]\}$. If the isotypical component

$$E_x^{[\sigma]} := i_\sigma\left(\mathrm{Hom}_H(W_\sigma, E_x) \otimes W_\sigma\right)$$

is nontrivial, then it is invariant under the subgroup $\widetilde{N}_{\alpha,[\sigma]} \subseteq \widetilde{N}_{[\sigma]}$ that leaves in addition the connected component X_α^H invariant; again, this subgroup has finite index in N. The isotypical components transform under $n \in N$ as

$$n : E_\alpha^{[\sigma]} \xrightarrow{\cong} E_{\overline{n}(\alpha)}^{[\sigma^n]},$$

where $\overline{n}$ denotes the residue class class of $n \in N$ in $N\diagup\widetilde{N}_{\alpha,[\sigma]}$. Then a decomposition of E is obtained by 'inducing up' the isotypical components $E_\alpha^{[\sigma]}$ from $\widetilde{N}_{\alpha,[\sigma]}$ to N. That is,

$$E_{\alpha,[\sigma]}^N = N \times_{\widetilde{N}_{\alpha,[\sigma]}} E_\alpha^{[\sigma]}$$

is a bundle containing $E_\alpha^{[\sigma]}\Big|_{X_\alpha^H}$. This is an N-bundle over $NX_\alpha^H \subseteq X^H$, and a similar bundle may be formed over each distinct NX_β^H, with $\beta \in \pi_0\left(X^H\right)$. Further, observe that since each bundle $E_{\alpha,[\sigma]}^N$ is an N-bundle over NX_α^H, it defines a unique G bundle $E_{\alpha,[\sigma]}^G$.

Definition 3.15. *The G-bundle $E_{\alpha,[\sigma]}^G$ over the submanifold X_α is called a* ***fine component*** *or the* ***fine component of*** *$E \to X$* ***associated to*** *$(\alpha,[\sigma])$.*

If $G\diagdown X$ is not connected, one must construct the fine components separately over each X_α. If E has finite rank, then E may be decomposed as a direct sum of distinct fine components over each X_α. In any case, $E_{\alpha,[\sigma]}^N$ is a finite direct sum of isotypical components over each X_α^H.

Definition 3.16. *The direct sum decomposition of $E|_{X_\alpha}$ into subbundles E^b that are fine components $E^G_{\alpha,[\sigma]}$ for some $[\sigma]$, written*

$$E|_{X_\alpha} = \bigoplus_b E^b \ ,$$

is called the ***refined isotypical decomposition*** *(or* ***fine decomposition****) of $E|_{X_\alpha}$.*

In the case where $G\diagdown X$ is connected, the group $\pi_0\left(N\diagup H\right)$ acts transitively on the connected components $\pi_0\left(X^H\right)$, and thus $X_\alpha = X$. We comment that if $[\sigma, W_\sigma]$ is an irreducible H-representation present in E_x with $x \in X^H_\alpha$, then $E^{[\sigma]}_x$ is a subspace of a distinct E^b_x for some b. The subspace E^b_x also contains $E^{[\sigma^n]}_x$ for every n such that $nX^H_\alpha = X^H_\alpha$.

Remark 3.17. *Observe that by construction, for $x \in X^H_\alpha$ the multiplicity and dimension of each $[\sigma]$ present in a specific E^b_x is independent of $[\sigma]$. Thus, $E^{[\sigma^n]}_x$ and $E^{[\sigma]}_x$ have the same multiplicity and dimension if $nX^H_\alpha = X^H_\alpha$.*

Remark 3.18. *The advantage of this decomposition over the isotypical decomposition is that each E^b is a G-bundle defined over all of X_α, and the isotypical decomposition may only be defined over X^H_α.*

Definition 3.19. *Now, let E be a G-equivariant vector bundle over X, and let E^b be a fine component as in Definition 3.15 corresponding to a specific component $X_\alpha = GX^H_\alpha$ of X relative to G. Suppose that another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E^b_y, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, and that W has a single component in the fine decomposition. Then we say that W is* ***adapted*** *to E^b.*

Lemma 3.20. *In the definition above, if another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E^b_y, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, then it follows that W has a single component in the fine decomposition and hence is adapted to E^b. Thus, the last phrase in the corresponding sentence in the above definition is superfluous.*

Proof. Suppose that we choose an equivalence class $[\sigma]$ of H-representations present in W_x, $x \in X^H_\alpha$. Let $[\sigma']$ be any other equivalence class; then, by hypothesis, there exists $n \in N$ such that $nX^H_\alpha = X^H_\alpha$ and $[\sigma'] = [\sigma^n]$. Then, observe that $nW^{[\sigma]}_x = W^{[\sigma^n]}_{nx} = W^{[\sigma^n]}_x$, with the last equality coming from the rigidity of irreducible H-representations. Thus, W is contained in a single fine component, and so it must have a single component in the fine decomposition. □

In what follows, we show that there are naturally defined finite-dimensional vector bundles that are adapted to any fine components. Once and for all, we enumerate the irreducible representations $\left\{\left[\rho_j, V_{\rho_j}\right]\right\}_{j=1,2,\dots}$ of G. Let $[\sigma, W_\sigma]$ be any irreducible H-representation. Let $G \times_H W_\sigma$ be the corresponding homogeneous vector bundle over the homogeneous space $G\diagup H$. Then the L^2-sections of this vector bundle decompose into irreducible G-representations. In particular, let $\left[\rho_{j_0}, V_{\rho_{j_0}}\right]$ be the equivalence class of irreducible representations that is present in $L^2\left(G\diagup H, G\times_H W_\sigma\right)$ and that has the lowest index j_0. Then Frobenius reciprocity implies

$$0 \neq \mathrm{Hom}_G\left(V_{\rho_{j_0}}, L^2\left(G\diagup H, G\times_H W_\sigma\right)\right) \cong \mathrm{Hom}_H\left(V_{\mathrm{Res}\left(\rho_{j_0}\right)}, W_\sigma\right),$$

so that the restriction of ρ_{j_0} to H contains the H-representation $[\sigma]$. Now, for a component X_α^H of X^H, with $X_\alpha = GX_\alpha^H$ its component in X relative to G, the trivial bundle

$$X_\alpha \times V_{\rho_{j_0}}$$

is a G-bundle (with diagonal action) that contains a nontrivial fine component $W_{\alpha,[\sigma]}$ containing $X_\alpha^H \times \left(V_{\rho_{j_0}}\right)^{[\sigma]}$.

Definition 3.21. *We call $W_{\alpha,[\sigma]} \to X_\alpha$ the **canonical isotropy G-bundle associated to** $(\alpha, [\sigma]) \in \pi_0\left(X^H\right) \times \widehat{H}$. Observe that $W_{\alpha,[\sigma]}$ depends only on the enumeration of irreducible representations of G, the irreducible H-representation $[\sigma]$ and the component X_α^H. We also denote the following positive integers associated to $W_{\alpha,[\sigma]}$:*

- $m_{\alpha,[\sigma]} = \dim \operatorname{Hom}_H\left(W_\sigma, W_{\alpha,[\sigma],x}\right) = \dim \operatorname{Hom}_H\left(W_\sigma, V_{\rho_{j_0}}\right)$ *(the **associated multiplicity**), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$ (see Remark 3.17).*
- $d_{\alpha,[\sigma]} = \dim W_\sigma$ *(the **associated representation dimension**), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.*
- $n_{\alpha,[\sigma]} = \frac{\operatorname{rank}\left(W_{\alpha,[\sigma]}\right)}{m_{\alpha,[\sigma]} d_{\alpha,[\sigma]}}$ *(the **inequivalence number**), the number of inequivalent representations present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.*

Remark 3.22. *Observe that $W_{\alpha,[\sigma]} = W_{\alpha',[\sigma']}$ if $[\sigma'] = [\sigma^n]$ for some $n \in N$ such that $nX_\alpha^H = X_{\alpha'}^H$.*

The lemma below follows immediately from Lemma 3.20.

Lemma 3.23. *Given any G-bundle $E \to X$ and any fine component E^b of E over some $X_\alpha = GX_\alpha^H$, there exists a canonical isotropy G-bundle $W_{\alpha,[\sigma]}$ adapted to $E^b \to X_\alpha$.*

An example of another foliated bundle over a component of a stratum M_j is the bundle defined as follows.

Definition 3.24. *Let $E \to M$ be any foliated vector bundle. Let $\Sigma_{\alpha_j} = \widehat{\pi}\left(p^{-1}\left(M_j\right)\right)$ be the corresponding component of the stratum relative to G on the basic manifold $\widehat{W}$ (see Section 3.3), and let $W^\tau \to \Sigma_{\alpha_j}$ be a canonical isotropy bundle (Definition 3.21). Consider the bundle $\widehat{\pi}^* W^\tau \otimes p^* E \to p^{-1}\left(M_j\right)$, which is foliated and basic for the lifted foliation restricted to $p^{-1}\left(M_j\right)$. This defines a new foliated bundle $E^\tau \to M_j$ by letting E_x^τ be the space of G-invariant sections of $\widehat{\pi}^* W^\tau \otimes p^* E$ restricted to $p^{-1}(x)$. We call this bundle **the W^τ-twist of** $E \to M_j$.*

4. Desingularization of the foliation

4.1. **Topological Desingularization.** Assume that $(M, \mathcal{F})$ is a Riemannian foliation, with principal stratum M_0 and singular strata $M_1, ..., M_r$ corresponding to isotropy types $[G_0]$, $[G_1]$, $[G_2]$, ..., $[G_r]$ on the basic manifold, as explained in Section 3.3. We will construct a new Riemannian foliation $(N, \mathcal{F}_N)$ that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the desingularization of M and is denoted $\widetilde{M}$. This process closely parallels the process of desingularizing a G-manifold, which is described in [14].

Recall the setup from Section 3.3. We are given $E \to M$, a foliated Hermitian vector bundle over M, and the bundle $\widehat{M} \xrightarrow{p} M$ is the bundle of ordered pairs (α, β) with structure group $G = O(q) \times U(k)$, with α a orthonormal transverse frame and β an orthonormal frame of E with respect to the Hermitian inner product on E, as in Section 3.1; in many cases the principal bundle may be reduced to a bundle with smaller structure group. The foliation $\mathcal{F}$ lifts to a foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$, and the lifted foliation is transversally parallelizable. We chose the natural metric on $\widehat{M}$ as in Section 3.1. By Molino theory ([48]), the leaf closures of $\widehat{\mathcal{F}}$ are diffeomorphic, have no holonomy, and form a Riemannian fiber bundle $\widehat{M} \xrightarrow{\widehat{\pi}} \widehat{W}$ over the basic manifold $\widehat{W}$, on which the group G acts by isometries. The G-orbits on $\widehat{W}$ and leaf closures of $(M, \mathcal{F})$ are identified via the correspondence

$$p\left(\widehat{\pi}^{-1}\left(G\text{–orbit on } \widehat{W}\right)\right) = \text{leaf closure of } (M, \mathcal{F}).$$

A sequence of modifications is used to construct N and $\widetilde{M} \subset N$. Let M_j be a minimal stratum. Let $T_\varepsilon(M_j)$ denote a tubular neighborhood of radius ε around M_j, with ε chosen sufficiently small so that all leaf closures in $T_\varepsilon(M_j) \setminus M_j$ correspond to isotropy types $[G_k]$, where $[G_k] < [G_j]$. Let

$$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M \setminus T_\varepsilon(M_j))$ smoothly along the boundary. Since the $T_\varepsilon(M_j)$ is saturated (a union of leaves), the foliation lifts to N^1. Note that the strata of the foliation $\mathcal{F}^1$ on N^1 correspond to strata in $M \setminus T_\varepsilon(M_j)$. If $M_k \cap (M \setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N_k^1 = (M_k \cap (M \setminus T_\varepsilon(M_j))) \cup_{(M_k \cap \partial T_\varepsilon(M_j))} (M_k \cap (M \setminus T_\varepsilon(M_j))).$$

Thus, $(N^1, \mathcal{F}^1)$ is a foliation with one fewer stratum than $(M, \mathcal{F})$, and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted $\widetilde{M}^1$ in N^1. One may radially modify metrics so that a bundle-like metric on $(M, \mathcal{F})$ transforms to a bundle-like metric on $(N^1, \mathcal{F}^1)$. In fact, N^1 is a branched double cover of M, branched over M_j. If the leaf closures of $(N^1, \mathcal{F}^1)$ correspond to a single orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If not, we repeat the process with the foliation $(N^1, \mathcal{F}^1)$ to produce a new Riemannian foliation $(N^2, \mathcal{F}^2)$ with two fewer strata than $(M, \mathcal{F})$ and that is a 4-fold branched cover of M. Again, $\widetilde{M}^2$ is a fundamental domain of $\widetilde{M}^1 \setminus \{\text{a minimal stratum}\}$, which is a fundamental domain of M with two strata removed. We continue until $(N, \mathcal{F}_N) = (N^r, \mathcal{F}^r)$ is a Riemannian foliation with all leaf closures corresponding to orbit type $[G_0]$ and is a 2^r-fold branched cover of M, branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^r$, which is a fundamental domain of M_0 in N.

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G-manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. The isotropy type corresponding to all leaf closures of $\widetilde{M_{\geq j}}$ is $[G_j]$, and $\widetilde{M_{\geq j} / \overline{\mathcal{F}}}$ is a smooth (open) manifold.

4.2. **Modification of the metric and differential operator.** We now more precisely describe the desingularization. If $(M, \mathcal{F})$ is equipped with a basic, transversally elliptic differential operator on sections of a foliated vector bundle over M, then this data may be pulled back to the desingularization $\widetilde{M}$. Given the bundle and operator over N^j, simply form the invertible double of the operator on N^{j+1}, which is the double of the manifold with boundary $N^j \setminus T_\varepsilon(\Sigma)$, where Σ is a minimal stratum on N^j.

Specifically, we modify the bundle-like metric radially so that there exists sufficiently small $\varepsilon > 0$ such that the (saturated) tubular neighborhood $B_{4\varepsilon}\Sigma$ of Σ in N^j is isometric to a ball of radius 4ε in the normal bundle $N\Sigma$. In polar coordinates, this metric is $ds^2 = dr^2 + d\sigma^2 + r^2 d\theta_\sigma^2$, with $r \in (0, 4\varepsilon)$, $d\sigma^2$ is the metric on Σ, and $d\theta_\sigma^2$ is the metric on $S(N_\sigma\Sigma)$, the unit sphere in $N_\sigma\Sigma$; note that $d\theta_\sigma^2$ is isometric to the Euclidean metric on the unit sphere. We simply choose the horizontal metric on $B_{4\varepsilon}\Sigma$ to be the pullback of the metric on the base Σ, the fiber metric to be Euclidean, and we require that horizontal and vertical vectors be orthogonal. We do not assume that the horizontal distribution is integrable. We that the metric constructed above is automatically bundle-like for the foliation.

Next, we replace r^2 with $f(r) = [\psi(r)]^2$ in the expression for the metric, where $\psi(r)$ is increasing, is a positive constant for $0 \le r \le \varepsilon$, and $\psi(r) = r$ for $2\varepsilon \le r \le 3\varepsilon$. Then the metric is cylindrical for $r < \varepsilon$.

In our description of the modification of the differential operator, we will need the notation for the (external) product of differential operators. Suppose that $F \hookrightarrow X \xrightarrow{\pi} B$ is a fiber bundle that is locally a metric product. Given an operator $A_{1,x}\colon \Gamma(\pi^{-1}(x), E_1) \to \Gamma(\pi^{-1}(x), F_1)$ that is locally given as a differential operator $A_1\colon \Gamma(F, E_1) \to \Gamma(F, F_1)$ and $A_2\colon \Gamma(B, E_2) \to \Gamma(B, F_2)$ on Hermitian bundles, we define the product

$$A_{1,x} * A_2\colon \Gamma(X, (E_1 \boxtimes E_2) \oplus (F_1 \boxtimes F_2)) \to \Gamma(X, (F_1 \boxtimes E_2) \oplus (E_1 \boxtimes F_2))$$

as the unique linear operator that satisfies locally

$$A_{1,x} * A_2 = \begin{pmatrix} A_1 \boxtimes \mathbf{1} & -\mathbf{1} \boxtimes A_2^* \\ \mathbf{1} \boxtimes A_2 & A_1^* \boxtimes \mathbf{1} \end{pmatrix}$$

on sections of

$$\begin{pmatrix} E_1 \boxtimes E_2 \\ F_1 \boxtimes F_2 \end{pmatrix}$$

of the form $\begin{pmatrix} u_1 \boxtimes u_2 \\ v_1 \boxtimes v_2 \end{pmatrix}$, where $u_1 \in \Gamma(F, E_1)$, $u_2 \in \Gamma(B, E_2)$, $v_1 \in \Gamma(F, F_1)$, $v_2 \in \Gamma(B, E_2)$. This coincides with the product in various versions of K-theory (see, for example, [1], [42, pp. 384ff]), which is used to define the Thom Isomorphism in vector bundles.

Let $D = D^+\colon \Gamma(N^j, E^+) \to \Gamma(N^j, E^-)$ be the given first order, transversally elliptic, $\mathcal{F}^j$-basic differential operator. Let Σ be a minimal stratum of N^j. We assume for the moment that Σ has codimension at least two. We modify the bundle radially so that the foliated bundle E over $B_{4\varepsilon}(\Sigma)$ is a pullback of the bundle $E|_\Sigma \to \Sigma$. We assume that near Σ, after a foliated homotopy D^+ can be written on $B_{4\varepsilon}(\Sigma)$ locally as the product

$$D^+ = (D_N * D_\Sigma)^+ , \tag{4.1}$$

where D_Σ is a transversally elliptic, basic, first order operator on the stratum $(\Sigma, \mathcal{F}|_\Sigma)$, and D_N is a basic, first order operator on $B_{4\varepsilon}(\Sigma)$ that is elliptic on the fibers. If r is the distance from Σ, we write D_N in polar coordinates as

$$D_N = Z\left(\nabla^E_{\partial_r} + \frac{1}{r} D^S\right)$$

where $Z = -i\sigma(D_N)(\partial_r)$ is a local bundle isomorphism and the map D^S is a purely first order operator that differentiates in the unit normal bundle directions tangent to $S_x\Sigma$.

We modify the operator D_N on each Euclidean fiber of $N\Sigma \xrightarrow{\pi} \Sigma$ by adjusting the coordinate r and function $\frac{1}{r}$ so that $D_N * D_\Sigma$ is converted to an operator on a cylinder;

see [14, Section 6.3.2] for the precise details. The result is a G-manifold $\widetilde{M^j}$ with boundary $\partial\widetilde{M^j}$, a G-vector bundle $\widetilde{E}^j$, and the induced operator $\widetilde{D}^j$, all of which locally agree with the original counterparts outside $B_\varepsilon(\Sigma)$. We may double $\widetilde{M^j}$ along the boundary $\partial\widetilde{M^j}$ and reverse the chirality of $\widetilde{E}^j$ as described in [8, Ch. 9]. Doubling produces a closed manifold N^j with foliation $\mathcal{F}^j$, a foliated bundle E^j, and a first-order transversally elliptic differential operator D^j. This process may be iterated until all leaf closures are principal. The case where some strata have codimension 1 is addressed in the following paragraphs.

We now give the definitions for the case when there is a minimal stratum Σ of codimension 1. Only the changes to the argument are noted. This means that the isotropy subgroup H corresponding to Σ contains a principal isotropy subgroup of index two. If r is the distance from Σ, then D_N has the form

$$D_N = Z\left(\nabla^E_{\partial_r} + \frac{1}{r}D^S\right) = Z\nabla^E_{\partial_r}$$

where $Z = -i\sigma(D_N)(\partial_r)$ is a local bundle isomorphism and the map $D^S = 0$.

In this case, there is no reason to modify the metric inside $B_\varepsilon(\Sigma)$. The "desingularization" of M along Σ is the manifold with boundary $\widetilde{M} = M \diagdown B_\delta(\Sigma)$ for some $0 < \delta < \varepsilon$; the singular stratum is replaced by the boundary $\partial\widetilde{M} = S_\delta(\Sigma)$, which is a two-fold cover of Σ and whose normal bundle is necessarily oriented (via ∂_r). The double M' is identical to the double of $\widetilde{M}$ along its boundary, and M' contains one less stratum.

4.3. **Discussion of operator product assumption.** We now explain specific situations that guarantee that, after a foliated homotopy, D^+ may be written locally as a product of operators as in (4.1) over the tubular neighborhhood $B_{4\varepsilon}(\Sigma)$ over a singular stratum Σ. This demonstrates that this assumption is not overly restrictive. We also emphasize that one might think that this assumption places conditions on the curvature of the normal bundle $N\Sigma$; however, this is not the case for the following reason. The condition is on the foliated homotopy class of the principal transverse symbol of D. The curvature of the bundle only effects the zeroth order part of the symbol. For example, if $Y \to X$ is any fiber bundle over a spin^c manifold X with fiber F, then a Dirac-type operator D on Y has the form $D = \partial_X * D_F + Z$, where D_F is a family of fiberwise Dirac-type operators, ∂_X is the spin^c Dirac operator on X, and Z is a bundle endomorphism.

First, we show that if D^+ is a transversal Dirac operator at points of Σ, and if either Σ is spin^c or its normal bundle $N\Sigma \to \Sigma$ is (fiberwise) spin^c, then it has the desired form. Moreover, we also remark that certain operators, like those resembling transversal de Rham operators, always satisfy this splitting condition with no assumptions on Σ.

Let $N\mathcal{F}$ be normal bundle of the foliation$\mathcal{F}_\Sigma = \mathcal{F}|_\Sigma$, and let $N\Sigma$ be the normal bundle of Σ in M. Then the principal transverse symbol of D^+ (evaluated at $\xi \in N^*_x\mathcal{F}_\Sigma \oplus N^*_x\Sigma$) at points $x \in \Sigma$ takes the form of a constant multiple of Clifford multiplication. That is, we assume there is an action c of $\mathbb{C}\mathrm{l}(N\mathcal{F}_\Sigma \oplus N\Sigma)$ on E and a Clifford connection ∇ on E such that the local expression for D is given by the composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*M) \xrightarrow{\mathrm{proj}} \Gamma(E \otimes (N^*\mathcal{F}_\Sigma \oplus N^*\Sigma)) \xrightarrow{\cong} \Gamma(E \otimes (N\mathcal{F}_\Sigma \oplus N\Sigma)) \xrightarrow{c} \Gamma(E).$$

The principal transverse symbol $\sigma\left(D^{+}\right)$ at $\xi_x \in T_x^*\Sigma$ is

$$\sigma\left(D^{+}\right)\left(\xi_x\right) = \sum_{j=1}^{q'} ic\left(\xi_x\right) : E_x^{+} \to E_x^{-}.$$

Suppose $N\Sigma$ is spin^c; then there exists a vector bundle $S = S^{+} \oplus S^{-} \to \Sigma$ that is an irreducible representation of $\mathbb{C}\mathrm{l}\left(N\Sigma\right)$ over each point of Σ, and we let $E^{\Sigma} = \mathrm{End}_{\mathbb{C}\mathrm{l}(N\Sigma)}\left(E\right)$ and have

$$E \cong S \widehat{\otimes} E^{\Sigma}$$

as a graded tensor product, such that the action of $\mathbb{C}\mathrm{l}\left(N\mathcal{F}_\Sigma \oplus N\Sigma\right) \cong \mathbb{C}\mathrm{l}\left(N\Sigma\right) \widehat{\otimes} \mathbb{C}\mathrm{l}\left(N\mathcal{F}_\Sigma\right)$ (as a graded tensor product) on E^{+} decomposes as

$$\begin{pmatrix} c\left(x\right) \otimes \mathbf{1} & -\mathbf{1} \otimes c\left(y\right)^* \\ \mathbf{1} \otimes c\left(y\right) & c\left(x\right)^* \otimes \mathbf{1} \end{pmatrix} : \begin{pmatrix} S^{+} \otimes E^{\Sigma +} \\ S^{-} \otimes E^{\Sigma -} \end{pmatrix} \to \begin{pmatrix} S^{-} \otimes E^{\Sigma +} \\ S^{+} \otimes E^{\Sigma -} \end{pmatrix}$$

(see [5], [42]). If we let the operator ∂^N denote the spin^c transversal Dirac operator on sections of $\pi^* S \to N\Sigma$, and let D_Σ be the transversal Dirac operator defined by the action of $\mathbb{C}\mathrm{l}\left(N\mathcal{F}_\Sigma\right)$ on E^{Σ}, then we have

$$D^{+} = \left(\partial^N * D_\Sigma\right)^{+}$$

up to zero$^{\mathrm{th}}$ order terms (coming from curvature of the fiber).

The same argument works if instead we have that the bundle $N\mathcal{F}_\Sigma \to \Sigma$ is spin^c. In this case a spin^c Dirac operator ∂^Σ on sections of a complex spinor bundle over Σ is transversally elliptic to the foliation $\mathcal{F}_\Sigma$, and we have a formula of the form

$$D^{+} = \left(D_N * \partial^\Sigma\right)^{+},$$

again up to zeroth order terms.

Even if $N\Sigma \to \Sigma$ and $N\mathcal{F}_\Sigma \to \Sigma$ are not spin^c, many other first order operators have splittings as in Equation (4.1). For example, if D^{+} is a transversal de Rham operator from even to odd forms, then D^{+} is the product of de Rham operators in the $N\Sigma$ and $N\mathcal{F}_\Sigma$ directions.

In [27], where a formula for the basic index is derived, the assumptions dictate that every isotropy subgroup is a connected torus, which implies that $N\Sigma \to \Sigma$ automatically carries a vertical almost complex structure and is thus spin^c, so that the splitting assumption is automatically satisfied in their paper as well.

5. The equivariant index theorem

We review some facts about equivariant index theory and in particular make note of [14, Theorem 9.2]. Suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold $\widehat{W}$. In the following sections of the paper, we will be particularly interested in the case where $\widehat{W}$ is the basic manifold associated to $(M, \mathcal{F})$ and $G = O\left(q\right)$. Let $E = E^{+} \oplus E^{-}$ be a graded, G-equivariant Hermitian vector bundle over $\widehat{W}$. We consider a first order G-equivariant differential operator $D = D^{+} : \Gamma\left(\widehat{W}, E^{+}\right) \to \Gamma\left(\widehat{W}, E^{-}\right)$ that is transversally elliptic, and let D^{-} be the formal adjoint of D^{+}.

The group G acts on $\Gamma\left(\widehat{W}, E^{\pm}\right)$ by $\left(gs\right)\left(x\right) = g \cdot s\left(g^{-1}x\right)$, and the (possibly infinite-dimensional) subspaces $\ker\left(D^{+}\right)$ and $\ker\left(D^{-}\right)$ are G-invariant subspaces. Let $\rho\colon G \to U\left(V_\rho\right)$

be an irreducible unitary representation of G, and let $\chi_\rho = \operatorname{tr}(\rho)$ denote its character. Let $\Gamma\left(\widehat{W},E^{\pm}\right)^{\rho}$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma\left(\widehat{W},E^{\pm}\right)$ that are unitarily equivalent to the representation ρ. It can be shown that the extended operators

$$\overline{D}_{\rho,s}\colon H^{s}\left(\Gamma\left(\widehat{W},E^{+}\right)^{\rho}\right)\to H^{s-1}\left(\Gamma\left(\widehat{W},E^{-}\right)^{\rho}\right)$$

are Fredholm and independent of s, so that each irreducible representation of G appears with finite multiplicity in $\ker D^{\pm}$ (see [14]). Let $a_\rho^{\pm}\in\mathbb{Z}_{\geq 0}$ be the multiplicity of ρ in $\ker\left(D^{\pm}\right)$.

The study of index theory for such transversally elliptic operators was initiated by M. Atiyah and I. Singer in the early 1970s ([1]). The virtual representation-valued index of D is given by

$$\operatorname{ind}^{G}(D):=\sum_{\rho}\left(a_\rho^{+}-a_\rho^{-}\right)[\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\operatorname{ind}^{\rho}(D):=a_\rho^{+}-a_\rho^{-}=\frac{1}{\dim V_\rho}\operatorname{ind}\left(D|_{\Gamma\left(\widehat{W},E^{+}\right)^{\rho}\to\Gamma\left(\widehat{W},E^{-}\right)^{\rho}}\right).$$

In particular, if ρ_0 is the trivial representation of G, then

$$\operatorname{ind}^{\rho_0}(D)=\operatorname{ind}\left(D|_{\Gamma\left(\widehat{W},E^{+}\right)^{G}\to\Gamma\left(\widehat{W},E^{-}\right)^{G}}\right),$$

where the superscript G implies restriction to G-invariant sections.

There is a clear relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\operatorname{ind}_g(D)$; the multiplicities determine the distributional index, and vice versa. The space $\Gamma\left(\widehat{W},E^{\pm}\right)^{\rho}$ is a subspace of the λ_ρ-eigenspace of C. The virtual character $\operatorname{ind}_g(D)$ is given by (see [1])

$$\begin{aligned}\operatorname{ind}_g(D) &:= \text{“}\operatorname{tr}\left(g|_{\ker D^{+}}\right)-\operatorname{tr}\left(g|_{\ker D^{-}}\right)\text{”}\\ &=\sum_{\rho}\operatorname{ind}^{\rho}(D)\,\chi_\rho(g).\end{aligned}$$

Note that the sum above does not in general converge, since $\ker D^{+}$ and $\ker D^{-}$ are in general infinite-dimensional, but it does make sense as a distribution on G. That is, if dg is the normalized, biinvariant Haar measure on G, and if $\phi=\beta+\sum c_\rho\chi_\rho\in C^{\infty}(G)$, with β orthogonal to the subspace of class functions on G, then

$$\begin{aligned}\operatorname{ind}_*(D)(\phi) &= \text{“}\int_G \phi(g)\ \overline{\operatorname{ind}_g(D)}\ dg\text{”}\\ &=\sum_{\rho}\operatorname{ind}^{\rho}(D)\int\phi(g)\ \overline{\chi_\rho(g)}\ dg=\sum_{\rho}\operatorname{ind}^{\rho}(D)\,c_\rho,\end{aligned}$$

an expression which converges because c_ρ is rapidly decreasing and $\operatorname{ind}^{\rho}(D)$ grows at most polynomially as ρ varies over the irreducible representations of G. From this calculation, we see that the multiplicities determine Atiyah's distributional index. Conversely, let $\alpha\colon G \to U(V_\alpha)$ be an irreducible unitary representation. Then

$$\operatorname{ind}_*(D)(\chi_\alpha) = \sum_\rho \operatorname{ind}^\rho(D) \int \chi_\alpha(g)\,\overline{\chi_\rho(g)}\,dg = \operatorname{ind}^\alpha D,$$

so that complete knowledge of the equivariant distributional index is equivalent to knowing all of the multiplicities $\operatorname{ind}^\rho(D)$. Because the operator $D|_{\Gamma(\widehat{W},E^+)^\rho \to \Gamma(\widehat{W},E^-)^\rho}$ is Fredholm, all of the indices $\operatorname{ind}^G(D)$, $\operatorname{ind}_g(D)$, and $\operatorname{ind}^\rho(D)$ depend only on the stable homotopy class of the principal transverse symbol of D.

The equivariant index theorem ([14, Theorem 9.2]) expresses $\operatorname{ind}^\rho(D)$ as a sum of integrals over the different strata of the action of G on $\widehat{W}$, and it involves the eta invariant of associated equivariant elliptic operators on spheres normal to the strata. The result is

$$\begin{aligned}
\operatorname{ind}^\rho(D) &= \int_{G\diagdown\widetilde{\widehat{W}_0}} A_0^\rho(x)\ \widetilde{|dx|}\ + \sum_{j=1}^r \beta\left(\Sigma_{\alpha_j}\right)\ ,\\
\beta\left(\Sigma_{\alpha_j}\right) &= \frac{1}{2\dim V_\rho} \sum_{b\in B} \frac{1}{n_b \operatorname{rank} W^b}\Big(-\eta\left(D_j^{S+,b}\right)\\
&\qquad + h\left(D_j^{S+,b}\right)\Big) \int_{G\diagdown\widetilde{\Sigma_{\alpha_j}}} A_{j,b}^\rho(x)\ \widetilde{|dx|}\ ,
\end{aligned}$$

(The notation is explained in [14]; the integrands $A_0^\rho(x)$ and $A_{j,b}^\rho(x)$ are the familar Atiyah-Singer integrands corresponding to local heat kernel supertraces of induced elliptic operators over closed manifolds.)

6. The basic index theorem

Suppose that E is a foliated $\mathbb{C}\mathrm{l}(Q)$ module with basic $\mathbb{C}\mathrm{l}(Q)$ connection ∇^E over a Riemannian foliation $(M,\mathcal{F})$. Let

$$D_b^E\colon \Gamma_b\left(E^+\right) \to \Gamma_b\left(E^-\right)$$

be the corresponding basic Dirac operator, with basic index $\operatorname{ind}_b\left(D_b^E\right)$.

In what follows, if U denotes an open subset of a stratum of $(M,\mathcal{F})$, U' denotes the desingularization of U very similar to that in Section 4, and $\widetilde{U}$ denotes the fundamental domain of U inside U'. We assume that near each component M_j of a singular stratum of $(M,\mathcal{F})$, D_b^E is homotopic (through basic, transversally elliptic operators) to the product $D_N * D_{M_j}$, where D_N is an $\mathcal{F}$-basic, first order differential operator on a tubular neighborhood of Σ_{α_j} that is elliptic and Zhas constant coefficients on the fibers and D_{M_j} is a global transversally elliptic, basic, first order operator on the Riemannian foliation $(M_j,\mathcal{F})$. In polar coordinates, the fiberwise elliptic operator D_N may be written

$$D_N = Z_j\left(\nabla^E_{\partial_r} + \frac{1}{r} D_j^S\right)\ ,$$

where r is the distance from M_j, where Z_j is a local bundle isometry (dependent on the spherical parameter), the map D_j^S is a family of purely first order operators that differentiates in directions tangent to the unit normal bundle of M_j.

Theorem 6.1. *(Basic Index Theorem for Riemannian foliations) Let M_0 be the principal stratum of the Riemannian foliation $(M,\mathcal{F})$, and let M_1, ... , M_r denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1]$, ... ,$[G_r]$ on the basic manifold. With notation as in the discussion above, we have*

$$\operatorname{ind}_b\left(D_b^E\right) = \int_{\widetilde{M_0 \diagup \overline{\mathcal{F}}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^{r}\beta(M_j)\ ,$$

$$\beta(M_j) = \frac{1}{2}\sum_{\tau}\frac{1}{n_\tau \operatorname{rank} W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right)+h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j \diagup \overline{\mathcal{F}}}} A^\tau_{j,b}(x)\ \widetilde{|dx|}\ ,$$

where the sum is over all components of singular strata and over all canonical isotropy bundles W^τ, only a finite number of which yield nonzero $A^\tau_{j,b}$, and where

1. *$A_{0,b}(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\widetilde{D_b^E}$ (a desingularization of D_b^E) on the quotient $\widetilde{M_0 \diagup \overline{\mathcal{F}}}$, where the bundle E is replaced by the space of basic sections of over each leaf closure;*
2. *$\eta\left(D_j^{S+,b}\right)$ and $h\left(D_j^{S+,b}\right)$ are the equivariant eta invariant and dimension of the equivariant kernel of the G_j-equivariant operator $D_j^{S+,b}$ (defined in a similar way as in* [14, formulas (6.3), (6.4), (6.7)]*);*
3. *$A^\tau_{j,b}(x)$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\left(\mathbf{1}\otimes D_{M_j}\right)'$ (blown-up and doubled from $\mathbf{1}\otimes D_{M_j}$, the twist of D_{M_j} by the canonical isotropy bundle W^τ from Definition 3.24) on the quotient $\widetilde{M_j \diagup \overline{\mathcal{F}}}$, where the bundle is replaced by the space of basic sections over each leaf closure; and*
4. *n_τ is the number of different inequivalent G_j-representation types present in a typical fiber of W^τ.*

Proof. Using Proposition 3.1, we have

$$\operatorname{ind}_b\left(D_b^E\right) = \operatorname{ind}\left(\mathcal{D}^G\right),$$

where $\mathcal{D}=\mathcal{D}^+$ is defined in (3.1). Let $\Sigma_{\alpha_1}, ..., \Sigma_{\alpha_r}$ denote the components of the strata of the basic manifold $\widehat{W}$ relative to the G-action corresponding to the components $M_1, ..., M_r$. Near each Σ_{α_j}, we write $\mathcal{D} = D_N * D^{\alpha_j}$, and write $D_N = Z_j\left(\nabla^E_{\partial_r}+\frac{1}{r}D_j^S\right)$ in polar coordinates. By the Invariant Index Theorem [14, Theorem 9.6], a special case of the Equivariant Index Theorem stated in the last section, we have

$$\operatorname{ind}\left(\mathcal{D}^G\right) = \int_{G\diagdown\widetilde{\widehat{W}_0}} A^G_0(x)\ \widetilde{|dx|}\ + \sum_{j=1}^{r}\beta\left(\Sigma_{\alpha_j}\right)\ ,$$

$$\beta\left(\Sigma_{\alpha_j}\right) = \frac{1}{2}\sum_{\tau\in B}\frac{1}{n_\tau \operatorname{rank} W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right)+h\left(D_j^{S+,\tau}\right)\right)\int_{G\diagdown\widetilde{\Sigma_{\alpha_j}}} A^G_{j,\tau}(x)\ \widetilde{|dx|}\ ,$$

where $\tau\in B$ only if W^τ corresponds to irreducible isotropy representations whose duals are present in E^{α_j}, the bundle on which D^{α_j} acts. First, $G\diagdown\widetilde{\widehat{W}_0} = \widetilde{M_0\diagup\overline{\mathcal{F}}}$, and $G\diagdown\widetilde{\Sigma_{\alpha_j}} = \widetilde{M_j\diagup\overline{\mathcal{F}}}$. By definition, $A^G_0(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\mathcal{D}'$ (blown-up and doubled from $\mathcal{D}$) on the quotient $G\diagdown\widehat{W}_0'$, where the bundle $\mathcal{E}\to\widehat{W}$ is replaced by the bundle of invariant sections of $\mathcal{E}$ over each orbit (corresponding to a point of $G\diagdown\widetilde{\widehat{W}_0}$). This is precisely the the space of basic sections of over the corresponding leaf closure (point of $\widetilde{M_0\diagup\overline{\mathcal{F}}}$), and the operator is the same as $\widetilde{D_b^E}$ by construction. Similarly, $A_{j,\tau}^G$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $(\mathbf{1}\otimes D^{\alpha_j})'$ (blown-up and doubled from $\mathbf{1}\otimes D^{\alpha_j}$, the twist of D^{α_j} by the canonical isotropy bundle $W^\tau\to\Sigma_{\alpha_j}$) on the quotient $G\diagdown\Sigma_{\alpha_j}'$, where the bundle is replaced by the space of invariant sections over each orbit. Again, this part of the formula is exactly that shown in the statement of the theorem. The quantities $-\eta\left(D_j^{S+,\tau}\right)+h\left(D_j^{S+,\tau}\right)$ in the equivariant and basic formulas are the same, since the spherical operator on the normal bundle to the stratum in the basic manifold is the same as the spherical operator defined on the normal bundle to the stratum of the Riemannian foliation. The theorem follows. z □

7. The representation-valued basic index theorem

In order to retain the complete information given by Atiyah's distributional index of the transversal differential operator $\mathcal{D}$, we need to consider the equivariant indices $\operatorname{ind}^\rho(\mathcal{D})$ associated to any irreducible representation ρ of $O(q)$.

Definition 7.1. *The* ***representation-valued basic index*** *of the transversal Dirac operator D_{tr}^E is defined as*

$$\operatorname{ind}_b^\rho\left(D_{\mathrm{tr}}^E\right)=\operatorname{ind}^\rho(\mathcal{D}).$$

Using [14, Theorem 9.2], we have the following result. The proof is no different than that of Theorem 6.1.

Theorem 7.2. *(Representation-valued Basic Index Theorem for Riemannian foliations) Let M_0 be the principal stratum of the Riemannian foliation $(M,\mathcal{F})$, and let $M_1,\ \dots\ ,M_r$ denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1],\ \dots\ ,[G_r]$ on the basic manifold. With notation as in the previous section, we have*

$$\operatorname{ind}_b^\rho\left(D_{\mathrm{tr}}^E\right)=\int_{\widetilde{M_0\diagup\overline{\mathcal{F}}}}A_0^\rho(x)\ \widetilde{|dx|}+\sum_{j=1}^r\beta(M_j)\ ,$$

$$\beta(M_j)=\frac{1}{2}\sum_\tau\frac{1}{n_\tau\mathrm{rank}\ W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right)+h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}}A_{j,\tau}^\rho(x)\ \widetilde{|dx|},$$

where $A_0^\rho(x)$ and $A_{j,\tau}^\rho(x)$ are the local Atiyah-Singer integrands of the operators induced on the leaf closure spaces by extracting the sections of type ρ from $\widetilde{M_0}$ and $\widetilde{M_j}$.

8. The basic index theorem for foliations given by suspension

One class of examples of Riemannian foliations are those constructed by suspensions. Let X be a closed manifold with fundamental group $\pi_1(X)$, which acts on the universal cover $\widetilde{X}$ by deck transformations. Let $\phi\colon\pi_1(X)\to\operatorname{Isom}(Y)$ be a homomorphism to the group of isometries of a closed Riemannian manifold Y. The suspension is defined to be

$$X\times_\phi Y=\widetilde{X}\times Y\diagup\sim,$$

where the equivalence relation is defined by $(x,y) \sim (x\cdot g^{-1}, \phi(g)\,y)$ for any $g \in \pi_1(X)$. The foliation $\mathcal{F}$ associated to this suspension is defined by the $\widetilde{X}$-parameter submanifolds, so that $T\mathcal{F}$ agrees with $T\widetilde{X}$ over each fundamental domain of $X \times_\phi Y$ in $\widetilde{X} \times Y$. This foliation is Riemannian, with transverse metric given by the metric on Y. A transversally-elliptic operator that preserves the foliation is simply an elliptic operator D^Y on Y that is G-equivariant, where $G = \phi(\pi_1(X)) \subset \operatorname{Isom}(Y)$. It follows that D^Y is also equivariant with respect to the action of the closure $\overline{G}$, a compact Lie group. Then we have that the basic index satisfies

$$\operatorname{ind}_b\left(D_b^Y\right) = \operatorname{ind}\left(\left(D^Y\right)^{\overline{G}}\right).$$

We wish to apply the basic index theorem to this example. Observe that the strata of the foliation $\mathcal{F}$ are determined by the strata of the $\overline{G}$-action on Y. Precisely, if $\Sigma_{\alpha_1}, ..., \Sigma_{\alpha_r}$ are the components of the strata of Y relative to $\overline{G}$, then each

$$M_j = \widetilde{X} \times \Sigma_{\alpha_j} \diagup \sim$$

is a component of a stratum of the foliation $(X \times_\phi Y, \mathcal{F})$. Similarly, the desingularizations of the foliation correspond exactly to the desingularizations of the group action in the Equivariant Index Theorem ([14]), applied to the $\overline{G}$ action on Y. By the basic index theorem,

$$\operatorname{ind}_b\left(D_b^Y\right) = \int_{\widetilde{M_0} \diagup \overline{\mathcal{F}}} A_{0,b}(x) \ \ \widetilde{|dx|} + \sum_{j=1}^r \beta(M_j) \quad = \int_{\widetilde{Y_0} \diagup \overline{G}} A_0^G(x) \ \ \widetilde{|dx|} + \sum_{j=1}^r \beta(M_j),$$

where $A_0^G(x)$ is the Atiyah-Singer integrand of the operator D^Y on the (blown up) quotient of the principal stratum of the $\overline{G}$-action, where the bundle is the space of invariant sections on the corresponding orbit. Similarly, the singular terms $\beta(M_j)$ are exactly the same as those in the Equivariant Index Theorem, applied to the $\overline{G}$ action on Y. Thus, the basic index theorem gives precisely the same formula as the Equivariant Index Theorem calculating the index $\operatorname{ind}\left(\left(D^Y\right)^{\overline{G}}\right)$.

We remark that in this particular case, the basic index may be calculated in an entirely different way, using the Atiyah-Segal fixed point formula for $\overline{G}$-equivariant elliptic operators (see [4]). Their formula is a formula for $\operatorname{ind}_g\left(D^Y\right)$, the difference of traces of the action of $g \in \overline{G}$ on $\ker\left(D^Y\right)$ and $\ker\left(D^{Y*}\right)$, and the answer is an integral $\int_{Y^g} \alpha_g$ of characteristic classes over the fixed point set $Y^g \subset Y$ of the element g. To extract the invariant part of this index, we would need to calculate

$$\operatorname{ind}\left(\left(D^Y\right)^{\overline{G}}\right) = \int_{\overline{G}} \operatorname{ind}_g\left(D^Y\right) dg = \int_{\overline{G}} \left(\int_{Y^g} \alpha_g\right) dg,$$

where dg is the normalized Haar measure. Since the fixed point set changes with g, the integral above could not be evaluated as above. However, if $\overline{G}$ is connected, we could use the Weyl integration formula to change the integral to an integral over a maximal torus T, and we could replace Y^g with the fixed point set Y^T, since for generic $g \in T$, $Y^g = Y^T$. Moreover, if G is not connected, one may construct a suspension Y' of the manifold on which a larger connected group G' acts such that $G' \diagdown Y' = G \diagdown Y$.

9. An example of transverse signature

In this section we give an example of a transverse signature operator that arises from an S^1 action on a 5-manifold. This is essentially a modification of an example from [1, pp. 84ff], and it illustrates the fact that the eta invariant term may be nonzero. Let Z^4 be a closed, oriented, 4-dimensional Riemannian manifold on which $\mathbb{Z}_p$ (p prime > 2) acts by isometries with isolated fixed points x_i, $i = 1, ..., N$. Let $M = Z^4 \times_{\mathbb{Z}_p} S^1$, where $\mathbb{Z}_p$ acts on S^1 by rotation by multiples of $\frac{2\pi}{p}$. Then S^1 acts on M, and $M \diagup S^1 \cong Z^4 \diagup \mathbb{Z}_p$.

Next, let D^+ denote the signature operator $d + d^*$ from self-dual to anti-self-dual forms on Z^4; this induces a transversally elliptic operator (also denoted by D^+). Then the S^1-invariant index of D^+ satisfies

$$\operatorname{ind}^{\rho_0}\left(D^+\right) = \operatorname{Sign}\left(M \diagup S^1\right) = \operatorname{Sign}\left(Z^4 \diagup \mathbb{Z}_p\right).$$

By the Invariant Index Theorem [14, Theorem 9.6] and the fact that the Atiyah-Singer integrand is the Hirzebruch L-polynomial $\frac{1}{3}p_1$,

$$\begin{aligned}\operatorname{ind}^{\rho_0}(D) = {} & \frac{1}{3}\int_{\widetilde{M} \diagup S^1} p_1 \\ & + \frac{1}{2}\sum_{j=1}^{N}\left(-\eta\left(D_j^{S+,\rho_0}\right) + h\left(D_j^{S+,\rho_0}\right)\right),\end{aligned}$$

where each D_i^{S+,ρ_0} is two copies of the boundary signature operator

$$B = (-1)^p\,(*d - d*)$$

on $2l$-forms ($l = 0, 1$) on the lens space $S^3 \diagup \mathbb{Z}_p$. We have $h\left(D_j^{S+,\rho_0}\right) = 2h(B) = 2$ (corresponding to constants), and in [3] the eta invariant is explicitly calculated to be

$$\eta\left(D_j^{S+,\rho_0}\right) = 2\eta(B) = -\frac{2}{p}\sum_{k=1}^{p-1}\cot\left(\frac{km_j\pi}{p}\right)\cot\left(\frac{kn_j\pi}{p}\right),$$

where the action of the generator ζ of $\mathbb{Z}_p$ on S^3 is

$$\zeta\cdot(z_1, z_2) = \left(e^{\frac{2m_j\pi i}{p}}z_1, e^{\frac{2n_j\pi i}{p}}z_2\right),$$

with $(m_j, p) = (n_j, p) = 1$. Thus,

$$\operatorname{Sign}\left(M \diagup S^1\right) = \frac{1}{3}\int_{\widetilde{Z^4} \diagup \mathbb{Z}_p} p_1 + \frac{1}{p}\sum_{j=1}^{N}\sum_{k=1}^{p-1}\cot\left(\frac{km_j\pi}{p}\right)\cot\left(\frac{kn_j\pi}{p}\right) + N$$

Note that in [1, pp. 84ff] it is shown that

$$\operatorname{Sign}\left(M \diagup S^1\right) = \frac{1}{3}\int_{Z^4 \diagup \mathbb{Z}_p} p_1 + \frac{1}{p}\sum_{j=1}^{N}\sum_{k=1}^{p-1}\cot\left(\frac{km_j\pi}{p}\right)\cot\left(\frac{kn_j\pi}{p}\right),$$

which demonstrates that

$$\frac{1}{3}\int_{Z^4 \diagup \mathbb{Z}_p} p_1 - \frac{1}{3}\int_{\widetilde{Z^4} \diagup \mathbb{Z}_p} p_1 = N,$$

illustrating the difference between the blowup $\widetilde{M}$ and the original M.

10. The Basic Euler characteristic

10.1. **The Basic Gauss-Bonnet Theorem.** Suppose that a smooth, closed manifold M is endowed with a smooth foliation $\mathcal{F}$.

In the theorem that follows, we express the basic Euler characteristic in terms of the ordinary Euler characteristic, which in turn can be expressed in terms of an integral of curvature. We extend the Euler characteristic notation $\chi(Y)$ for Y any open (noncompact without boundary) or closed (compact without boundary) manifold to mean

$$\chi(Y) = \begin{array}{ll} \chi(Y) & \text{if } Y \text{ is closed} \\ \chi(\text{1-point compactification of } Y) - 1 & \text{if } Y \text{ is open.} \end{array}$$

Also, if $\mathcal{L}$ is a flat foliated line bundle over a Riemannian foliation $(X, \mathcal{F})$, we define the basic Euler characteristic $\chi(X, \mathcal{F}, \mathcal{L})$ as before, using the basic cohomology groups with coefficients in the line bundle $\mathcal{L}$.

Theorem 10.1. *(Basic Gauss-Bonnet Theorem, announced in* [53]*) Let $(M, \mathcal{F})$ be a Riemannian foliation. Let M_0,..., M_r be the strata of the Riemannian foliation $(M, \mathcal{F})$, and let $\mathcal{O}_{M_j \diagup \overline{\mathcal{F}}}$ denote the orientation line bundle of the normal bundle to $\overline{\mathcal{F}}$ in M_j. Let L_j denote a representative leaf closure in M_j. With notation as above, the basic Euler characteristic satisfies*

$$\chi(M, \mathcal{F}) = \sum_j \chi\left(M_j \diagup \overline{\mathcal{F}}\right) \chi\left(L_j, \mathcal{F}, \mathcal{O}_{M_j \diagup \overline{\mathcal{F}}}\right).$$

Remark 10.2. *In* [27, Corollary 1]*, they show that in special cases the only term that appears is one corresponding to a most singular stratum.*

10.1.1. *Proof using the basic Hopf index theorem.* In this section, we prove the basic Gauss-Bonnet Theorem using the Hopf index theorem for Riemannian foliations ([6]).

To find a topological formula for the basic index, we first construct a basic, normal, $\mathcal{F}'$-nondegenerate vector field V on $(M, \mathcal{F})$ and then compute the basic Euler characteristic from this information. The formula from the main theorem in [6] is

$$\chi(M, \mathcal{F}) = \sum_{L \text{ critical}} \operatorname{ind}(V, L)\, \chi(L, \mathcal{F}, \mathcal{O}_L).$$

We construct the vector field as follows. First, starting with $i = 1$ (where the holonomy is largest, where $M_i \diagup \overline{\mathcal{F}}$ is a closed manifold), we triangulate $\overline{M_i} \diagup \overline{\mathcal{F}} \cong \widehat{\overline{W}(G_i)} \diagup G$, without changing the triangulation of $\left(\overline{M_i} \diagup \overline{\mathcal{F}}\right) \setminus M_i \diagup \overline{\mathcal{F}}$ (to construct the triangulation, we may first apply the exponential map of M_i to the normal space to a specific leaf closure of M_i and extend the geodesics to the cut locus, and so on). The result is a triangulation of $M \diagup \overline{\mathcal{F}}$ that restricts to a triangulation of each $\overline{M_i} \diagup \overline{\mathcal{F}}$. Next, we assign the value 0 to each vertex of the triangulation and the value k to a point on the interior of each k-cell, and we smoothly extend this function to a smooth basic Morse function on all of M whose only critical leaf closures are each of the points mentioned above. The gradient of this function is a a basic, normal, $\mathcal{F}'$-nondegenerate vector field V on M. Thus, letting L_k denote a leaf closure corresponding to the value k,

$$\chi(M, \mathcal{F}) = \sum_{L \text{ critical}} \operatorname{ind}(V, L)\, \chi(L, \mathcal{F}, \mathcal{O}_L)$$

$$
\begin{aligned}
&= \sum_k \sum_{L_k} (-1)^k \chi(L_k, \mathcal{F}, \mathcal{O}_L) \\
&= \sum_i \chi\left(M_i \diagup \overline{\mathcal{F}}\right) \chi(L_i, \mathcal{F}, \mathcal{O}_{L_i}) \\
&= \sum_i \chi\left(M_i \diagup \overline{\mathcal{F}}\right) \chi\left(L_i, \mathcal{F}, \mathcal{O}_{M_i \diagup \overline{\mathcal{F}}}\right)
\end{aligned}
$$

where L_i denotes a representative leaf closure of M_i, and $\mathcal{O}_{L_i}$ denotes its "negative direction orientation bundle", which by the definition of the vector field is isomorphic to the orientation bundle $\mathcal{O}_{M_i \diagup \overline{\mathcal{F}}}$ of $T\left(M_i \diagup \overline{\mathcal{F}}\right)$.

10.1.2. *Proof using the Basic Index Theorem.* In this section, we prove the basic Gauss-Bonnet Theorem using the Basic Index Theorem (Theorem 6.1).

As explained in Section 2.5 we wish to compute $\mathrm{ind}_b(D_b') = \mathrm{ind}_b(D_b)$, with

$$D_b' = d + \delta_b;\ D_b = D_b' - \frac{1}{2}(\kappa_b \wedge + \kappa_b \lrcorner).$$

Let M_0 be the principal stratum of the Riemannian foliation $(M, \mathcal{F})$, and let $M_1, \ldots, M_r$ denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1]$, $\ldots$, $[G_r]$ on the basic manifold. At each M_j, we may write the basic de Rham operator (up to lower order perturbations) as

$$D_b = D_{N_j} * D_{M_j}\ ,$$

where D_{N_j} is in fact the de Rham operator on the vertical forms, and D_{M_j} is the basic de Rham operator on $\left(M_j\ , \mathcal{F}|_{M_j}\right)$. Further, the spherical operator D_j^S in the main theorem is simply

$$D_j^S = -c(\partial_r)(d + d^*)^S,\ c(\partial_r) = dr \wedge - dr \lrcorner\ ,$$

where $(d + d^*)^S$ is a vector-valued de Rham operator on the sphere (normal to M_j) and r is the radial distance from M_j . We performed a similar calculation in [14, Section 10.2], and the results are that $\eta\left(D_j^{S+,\sigma}\right) = 0$ for all G_j-representation types $[\sigma]$ and

$$
h\left(D_j^{S+,\sigma}\right) = \begin{cases} 2 & \text{if } \sigma = \mathbf{1} \text{ and } G_j \text{ preserves orientation} \\ 1 & \text{if } \sigma = \mathbf{1} \text{ and } G_j \text{ does not preserve orientation} \\ 1 & \text{if } \sigma = \xi_{G_j} \text{ and } G_j \text{ does not preserve orientation} \\ 0 & \text{otherwise.} \end{cases} \tag{10.1}
$$

Here, if some elements of G_j reverse orientation of the normal bundle, then ξ_{G_j} denotes the relevant one-dimensional representation of G_j as ± 1. The orientation line bundle $\mathcal{O}_{M_i \diagup \overline{\mathcal{F}}} \to M_j$ of the normal bundle to M_j is a pointwise representation space for the representation ξ_{G_j}. After pulling back to and pushing forward to the basic manifold, it is the canonical isotropy G-bundle W^b corresponding to $\left(j, \left[\xi_{G_j}\right]\right)$. We may also take it to be a representation bundle for the trivial G_j-representation $\mathbf{1}$ (although the trivial line bundle is the canonical one). The Basic Index Theorem takes the form

$$\mathrm{ind}_b\left(D_b^E\right) = \int_{\widetilde{M_0 \diagup \overline{\mathcal{F}}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^r \beta(M_j)$$

$$\beta(M_j) = \frac{1}{2}\sum_j \left(h\left(D_j^{S+,\xi_{G_j}}\right) + +h\left(D_j^{S+,\mathbf{1}}\right)\right)\int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\ \widetilde{|dx|}$$

$$= \sum_j \int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\ \widetilde{|dx|}.$$

We rewrite $\int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\ \widetilde{|dx|}$ as $\int_{\widetilde{M_j}} K_j\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\ \widetilde{|dx|}$ before taking it to the quotient. We see that $K_j\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)$ is the Gauss-Bonnet integrand on the desingularized stratum $\widetilde{M_j}$, restricted to $\mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}$-twisted basic forms. The result is the relative Euler characteristic$\chi\left(L_j, \mathcal{F}, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)$

$$\int_{\widetilde{M_j}} K_j\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\ \widetilde{|dx|} = \chi\left(\overline{M_j}, \text{lower strata}, \mathcal{F}, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right),$$

Here, the relative basic Euler characteristic is defined for X a closed subset of a manifold Y as $\chi(Y, X, \mathcal{F}, \mathcal{V}) = \chi(Y, \mathcal{F}, \mathcal{V}) - \chi(X, \mathcal{F}, \mathcal{V})$, which is also the alternating sum of the dimensions of the relative basic cohomology groups with coefficients in a complex vector bundle $\mathcal{V} \to Y$. Since M_j is a fiber bundle over $M_j\diagup\overline{\mathcal{F}}$ with fiber L_j (a representative leaf closure), we have

$$\int_{\widetilde{M_j}} K_j\left(x, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\ \widetilde{|dx|} = \chi\left(L_j, \mathcal{F}, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)\chi\left(\overline{M_j}\diagup\overline{\mathcal{F}}, \text{lower strata}\diagup\overline{\mathcal{F}}\right),$$

by the formula for the Euler characteristic on fiber bundles, which extends naturally to the current situation. The Basic Gauss-Bonnet Theorem follows.

10.1.3. *The representation-valued basic Euler characteristic.* Using the Representation-valued Basic Index Theorem (Theorem 7.2), we may use the arguments in the previous section to derive a formula for the basic Euler characteristic of basic forms twisted by a representation of $O(q)$. Since the proof is nearly the same, we simply state the result.

Theorem 10.3. *(Representation-valued Basic Gauss-Bonnet Theorem) Let $(M, \mathcal{F})$ be a Riemannian foliation. Let M_0,..., M_r be the strata of the Riemannian foliation $(M, \mathcal{F})$, and let $\mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}$ denote the orientation line bundle of the normal bundle to $\overline{\mathcal{F}}$ in M_j. Let L_j denote a representative leaf closure in M_j. For $(X, \mathcal{F}_X)$ a Riemannian foliation of codimension q, let $\chi^\rho(X, \mathcal{F}_X, \mathcal{V})$ denote the index of the basic de Rham operator twisted by a representation $\rho\colon O(q) \to U(V_\rho)$ with values in the flat line bundle$\mathcal{V}$. Then the basic Euler characteristic satisfies*

$$\chi^\rho(M, \mathcal{F}) = \sum_j \chi\left(M_j\diagup\overline{\mathcal{F}}\right)\chi^\rho\left(L_j, \mathcal{F}, \mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right).$$

10.2. **Examples of the basic Euler characteristic.** In addition to the examples in this section, we refer the reader to [29], where in some nontaut Riemannian foliations, the basic Euler characteristic and basic cohomology groups and twisted basic cohomology groups are computed using the theorems in this paper.

The first example is a codimension 2 foliation on a 3-manifold. Here, $O(2)$ acts on the basic manifold, which is homeomorphic to a sphere. In this case, the principal orbits have isotropy type $(\{e\})$, and the two fixed points obviously have isotropy type $(O(2))$. In this example, the isotropy types correspond precisely to the infinitesimal holonomy groups.

Example 10.4. *(This example is taken from* [51] *and* [55]*.) Consider the one dimensional foliation obtained by suspending an irrational rotation on the standard unit sphere S^2. On S^2 we use the cylindrical coordinates (z,θ), related to the standard rectangular coordinates by $x' = \sqrt{(1-z^2)}\cos\theta$, $y' = \sqrt{(1-z^2)}\sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and let the three–manifold $M = S^2 \times [0,1] / \sim$, where $(z,\theta,0) \sim (z,\theta+\alpha,1)$. Endow M with the product metric on $T_{z,\theta,t}M \cong T_{z,\theta}S^2 \times T_t\mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta} = \cup_{n\in\mathbb{Z}}\{z\}\times\{\theta+\alpha\}\times[0,1]$ (not unique in θ). The leaf closures $\overline{L}_z$ for $|z|<1$ are two dimensional, and the closures corresponding to the poles ($z=\pm1$) are one dimensional.*
The stratification of $(M,\mathcal{F})$ is $M(H_1)\coprod M(H_2)$, where $M(H_1)$ is the union of the two "polar" leaves ($z=\pm1$), and $M(H_2)$ is the complement of $M(H_1)$. Note that each orientation bundle $\mathcal{O}_{M(H_i)\diagup\overline{\mathcal{F}}}$ is trivial. Next, $\chi\left(M(H_2)\diagup\overline{\mathcal{F}}\right) = \chi$ (open interval) $=-1$, and $\chi\left(M(H_1)\diagup\overline{\mathcal{F}}\right) = \chi$ (disjoint union of two points) $=2$. Observe that $\chi\left(L_1,\mathcal{F},\mathcal{O}_{M(H_1)\diagup\overline{\mathcal{F}}}\right) = \chi(L_1,\mathcal{F}) = \chi(S^1,S^1) = 1$. However, $\chi\left(L_2,\mathcal{F},\mathcal{O}_{M(H_2)\diagup\overline{\mathcal{F}}}\right) = \chi(L_2,\mathcal{F}) = 0$, since every such leaf closure is a flat torus, on which the foliation restricts to be the irrational flow and since the vector field ∂_θ is basic, nonsingular, and orthogonal to the foliation on this torus. By our theorem, we conclude that

$$\begin{aligned}\chi(M,\mathcal{F}) &= \sum_i \chi\left(M(H_i)\diagup\overline{\mathcal{F}}\right)\chi\left(L_i,\mathcal{F},\quad \mathcal{O}_{M(H_i)\diagup\overline{\mathcal{F}}}\right)\\ &= 2\cdot 1 + (-1)\cdot 0 = 2.\end{aligned}$$

We now directly calculate the Euler characteristic of this foliation. Since the foliation is taut, the standard Poincare duality works [35] [36] *, and $H_b^0(M) \cong H_b^2(M) \cong \mathbb{R}$. It suffices to check the dimension h^1 of the cohomology group $H_b^1(M)$. Then the basic Euler characteristic is $\chi(M,\mathcal{F}) = 1 - h^1 + 1 = 2 - h^1$. Smooth basic functions are of the form $f(z)$, where $f(z)$ is smooth in z for $-1<z<1$ and is of the form $f(z) = f_1(1-z^2)$ near $z=1$ for a smooth function f_1 and is of the form $f(z) = f_2(1-z^2)$ near $z=-1$ for a smooth function f_2. Smooth basic one forms are of the form $\alpha = g(z)\,dz + k(z)\,d\theta$, where $g(z)$ and $k(z)$ are smooth functions for $-1<z<1$ and satisfy*

$$\begin{aligned} g(z) &= g_1\left(1-z^2\right) \text{ and} \\ k(z) &= \left(1-z^2\right)k_1\left(1-z^2\right) \end{aligned} \tag{10.2}$$

near $z=1$ and

$$\begin{aligned} g(z) &= g_2\left(1-z^2\right) \text{ and} \\ k(z) &= \left(1-z^2\right)k_2\left(1-z^2\right) \end{aligned}$$

near $z=-1$ for smooth functions g_1, g_2, k_1, k_2 . A simple calculation shows that $\ker d^1 = \operatorname{im} d^0$, so that $h^1 = 0$. Thus, $\chi(M,\mathcal{F}) = 2$. This example shows that the orbit space can be dimension 1 (odd) and yet have nontrivial index.

The next example is a codimension 3 Riemannian foliation for which all of the infinitesimal holonomy groups are trivial; moreover, the leaves are all simply connected. There are leaf closures of codimension 2 and codimension 1. The codimension 1 leaf closures correspond to isotropy type (e) on the basic manifold, and the codimension 2 leaf closures correspond to an isotropy type $(O(2))$ on the basic manifold. In some sense, the isotropy type measures the holonomy of the leaf closure in this case.

Example 10.5. *This foliation is a suspension of an irrational rotation of S^1 composed with an irrational rotation of S^2 on the manifold $S^1 \times S^2$. As in Example 10.4, on S^2 we use the cylindrical coordinates (z, θ), related to the standard rectangular coordinates by $x' = \sqrt{(1-z^2)}\cos\theta$, $y' = \sqrt{(1-z^2)}\sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and let β be any irrational number. We consider the four–manifold $M = S^2 \times [0,1] \times [0,1] / \sim$, where $(z,\theta,0,t) \sim (z,\theta,1,t)$, $(z,\theta,s,0) \sim (z,\theta+\alpha, s+\beta \mod 1, 1)$. Endow M with the product metric on $T_{z,\theta,s,t}M \cong T_{z,\theta}S^2 \times T_s\mathbb{R} \times T_t\mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta,s} = \cup_{n\in\mathbb{Z}} \{z\} \times \{\theta+\alpha\} \times \{s+\beta\} \times [0,1]$ (not unique in θ or s). The leaf closures $\overline{L}_z$ for $|z|<1$ are three–dimensional, and the closures corresponding to the poles ($z = \pm 1$) are two–dimensional. The basic forms in the various dimensions are:*

$$\begin{aligned}
\Omega_b^0 &= \{f(z)\} \\
\Omega_b^1 &= \left\{g_1(z)\,dz + \left(1-z^2\right) g_2(z)d\theta + g_3(z)\,ds\right\} \\
\Omega_b^2 &= \left\{h_1(z)\,dz\wedge d\theta + \left(1-z^2\right) h_2(z)d\theta\wedge ds + h_3(z)\,dz\wedge ds\right\} \\
\Omega_b^3 &= \{k(z)\,dz\wedge d\theta\wedge ds\},
\end{aligned}$$

where all of the functions above are smooth in a neighborhood of $[0,1]$. An elementary calculation shows that $h^0 = h^1 = h^2 = h^3 = 1$, so that $\chi(M,\mathcal{F}) = 0$.

We now compute the basic Euler characteristic using our theorem. The stratification of $(M,\mathcal{F})$ is $M(H_1) \coprod M(H_2)$, where $M(H_1)$ is the union of the two "polar" leaf closures ($z = \pm 1$), and $M(H_2)$ is the complement of $M(H_1)$. Note that each orientation bundle $\mathcal{O}_{M(H_i)/\overline{\mathcal{F}}}$ is trivial. Next, $\chi\left(M(H_2)\diagup\overline{\mathcal{F}}\right) = \chi$ (open interval) $= -1$, and $\chi\left(M(H_1)\diagup\overline{\mathcal{F}}\right) = \chi$ (disjoint union of two points) $= 2$.
Observe that $\chi\left(L_1, \mathcal{F}, \mathcal{O}_{M(H_1)\diagup\overline{\mathcal{F}}}\right) = \chi(L_1,\mathcal{F}) = 0$, since this is a taut, codimension-1 foliation. Also, $\chi\left(L_2, \mathcal{F}, \mathcal{O}_{M(H_2)\diagup\overline{\mathcal{F}}}\right) = \chi(L_2,\mathcal{F}) = 1-2+1 = 0$, since the basic forms restricted to L_2 consist of the span of the set of closed forms $\{1, d\theta, ds, d\theta\wedge ds\}$. Thus,

$$\begin{aligned}
\chi(M,\mathcal{F}) &= \sum_i \chi\left(M(H_i)\diagup\overline{\mathcal{F}}\right)\chi\left(L_i,\mathcal{F},\mathcal{O}_{M(H_i)\diagup\overline{\mathcal{F}}}\right) \\
&= 2\cdot 0 + (-1)\cdot 0 = 0,
\end{aligned}$$

as we have already seen.

Note that taut foliations of odd codimension will always have a zero Euler characteristic, by Poincare duality. Open Question: will these foliations always have a zero basic index?

The following example is a codimension two transversally oriented Riemannian foliation in which all the leaf closures have codimension one. The leaf closure foliation is not transversally orientable, and the basic manifold is a flat Klein bottle with an $O(2)$–action. The two leaf closures with $\mathbb{Z}_2$ holonomy correspond to the two orbits of type $(\mathbb{Z}_2)$, and the other orbits have trivial isotropy.

Example 10.6. *This foliation is the suspension of an irrational rotation of the flat torus and a $\mathbb{Z}_2$–action. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z}*\mathbb{Z}$, the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$*

be the universal cover. We define $M = \widetilde{X} \times S^1 \times S^1 \diagup \pi_1(X)$*, where* $\pi_1(X)$ *acts by deck transformations on* $\widetilde{X}$ *and by* $\alpha(\theta, \phi) = (2\pi - \theta, 2\pi - \phi)$ *and* $\beta(\theta, \phi) = \left(\theta, \phi + \sqrt{2}\pi\right)$ *on* $S^1 \times S^1$*. We use the standard product–type metric. The leaves of* $\mathcal{F}$ *are defined to be sets of the form* $\left\{(x, \theta, \phi)_\sim \,|\, x \in \widetilde{X}\right\}$*. Note that the foliation is transversally oriented. The leaf closures are sets of the form*

$$\overline{L}_\theta = \left\{(x, \theta, \phi)_\sim \,|\, x \in \widetilde{X}, \phi \in [0, 2\pi]\right\} \bigcup \left\{(x, 2\pi - \theta, \phi)_\sim \,|\, x \in \widetilde{X}, \phi \in [0, 2\pi]\right\}$$

The basic forms are:

$$\begin{aligned} \Omega_b^0 &= \{f(\theta)\} \\ \Omega_b^1 &= \{g_1(\theta)\, d\theta + g_2(\theta) d\phi\} \\ \Omega_b^2 &= \{h(\theta) d\theta \wedge d\phi\}, \end{aligned}$$

where the functions are smooth and satisfy

$$\begin{aligned} f(2\pi - \theta) &= f(\theta) \\ g_i(2\pi - \theta) &= -g_i(\theta) \\ h(2\pi - \theta) &= h(\theta). \end{aligned}$$

A simple argument shows that $h^0 = h^2 = 1$ *and* $h^1 = 0$*. Thus,* $\chi(M, \mathcal{F}) = 2$*. The basic manifold* $\widehat{W}$ *is an* $O(2)$*–manifold, defined by* $\widehat{W} = [0, \pi] \times S^1 \diagup \sim$ *, where the circle has length* 1 *and* $(\theta = 0 \text{ or } \pi, \gamma) \sim (\theta = 0 \text{ or } \pi, -\gamma)$*. This is a Klein bottle, since it is the connected sum of two projective planes.* $O(2)$ *acts on* $\widehat{W}$ *via the usual action on* S^1*.*

Next, we compute the basic Euler characteristic using our theorem. The stratification of $(M, \mathcal{F})$ *is* $M(H_1) \coprod M(H_2)$*, where* $M(H_1)$ *is the union of the two leaf closures* $\theta_2 = 0$ *and* $\theta_2 = \pi$*, and* $M(H_2)$ *is the complement of* $M(H_1)$*. Note that the orientation bundle* $\mathcal{O}_{M(H_2)\diagup\overline{\mathcal{F}}}$ *is trivial since an interval is orientable, and* $\mathcal{O}_{M(H_1)\diagup\overline{\mathcal{F}}}$ *is trivial even though those leaf closures are not transversally oriented (since the points are oriented!). Next,*

$$\chi\left(M(H_2)\diagup\overline{\mathcal{F}}\right) = \chi(\text{open interval}) = -1,$$

and

$$\chi\left(M(H_1)\diagup\overline{\mathcal{F}}\right) = \chi(\text{disjoint union of two points}) = 2.$$

Observe that $\chi\left(L_2, \mathcal{F}, \mathcal{O}_{M(H_2)\diagup\overline{\mathcal{F}}}\right) = \chi(L_2, \mathcal{F}) = 0$*, since each representative leaf* L_2 *is a taut (since it is a suspension), codimension* 1 *foliation, and thus ordinary Poincare duality holds* ([58],[49]*):* $\dim H_B^0(L_2, \mathcal{F}) = \dim H_B^1(L_2, \mathcal{F}) = 1$*. On the other hand,* $\chi\left(L_1, \mathcal{F}, \mathcal{O}_{M(H_1)\diagup\overline{\mathcal{F}}}\right) = \chi(L_1, \mathcal{F}) = 1$*, since each such leaf closure has* $\dim H_B^0(L_1, \mathcal{F}) = 1$ *but* $\dim H_B^1(L_1, \mathcal{F}) = 0$ *since there are no basic one-forms. By our theorem, we conclude that*

$$\begin{aligned} \chi(Y, \mathcal{F}) &= \sum_i \chi\left(M(H_i)\diagup\overline{\mathcal{F}}\right) \chi\left(L_i, \mathcal{F},\ \mathcal{O}_{M(H_i)\diagup\overline{\mathcal{F}}}\right) \\ &= 2 \cdot 1 + (-1) \cdot 0 = 2, \end{aligned}$$

as we found before by direct calculation.

The next example is a codimension two Riemannian foliation with dense leaves, such that some leaves have holonomy but most do not. The basic manifold is a point, the fixed point set of the $O(2)$ action. The isotropy group $O(2)$ measures the holonomy of some of the leaves contained in the leaf closure.

Example 10.7. *This Riemannian foliation is a suspension of a pair of rotations of the sphere S^2. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$, that is the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M = \widetilde{X} \times S^2 \diagup \pi_1(X)$. The group $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by rotations on S^2 in the following ways. Thinking of S^2 as imbedded in $\mathbb{R}^3$, let α act by an irrational rotation around the z–axis, and let β act by an irrational rotation around the x–axis. We use the standard product–type metric. As usual, the leaves of $\mathcal{F}$ are defined to be sets of the form $\left\{(x,v)_\sim \,|\, x \in \widetilde{X}\right\}$. Note that the foliation is transversally oriented, and a generic leaf is simply connected and thus has trivial holonomy. Also, the every leaf is dense. The leaves $\{(x,(1,0,0))_\sim\}$ and $\{(x,(0,0,1))_\sim\}$ have nontrivial holonomy; the closures of their infinitesimal holonomy groups are copies of $SO(2)$. Thus, a leaf closure in $\widehat{M}$ covering the leaf closure M has structure group $SO(2)$ and is thus all of $\widehat{M}$, so that $\widehat{W}$ is a point. The only basic forms are constants and 2 forms of the form CdV, where C is a constant and dV is the volume form on S^2. Thus $h^0 = h^2 = 1$ and $h^1 = 0$, so that $\chi(M, \mathcal{F}) = 2$.*

Our theorem in this case, since there is only one stratum, is

$$\begin{aligned}
\chi(M,\mathcal{F}) &= \sum_i \chi\left(M(H_i)\diagup\overline{\mathcal{F}}\right)\chi\left(L_i, \mathcal{F}, \quad \mathcal{O}_{M(H_i)\diagup\overline{\mathcal{F}}}\right) \\
&= \chi(\textit{point})\,\chi(M,\mathcal{F}) \\
&= \chi(M,\mathcal{F}),
\end{aligned}$$

which is perhaps not very enlightening.

The following example is a codimension two Riemannian foliation that is not taut. This example is in [16].

Example 10.8. *Consider the flat torus $T^2 = \mathbb{R}^2 \diagup \mathbb{Z}^2$. Consider the map $F\colon T^2 \to T^2$ defined by*

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \bmod 1$$

Let $M = [0,1] \times T^2 \diagup \sim$, where $(0,a) \sim (1, F(a))$. Let v, v' be orthonormal eigenvectors of the matrix above, corresponding to the eigenvalues $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$, respectively. Let the linear foliation $\mathcal{F}$ be defined by the vector v' on each copy of T^2. Notice that every leaf is simply connected and that the leaf closures are of the form $\{t\} \times T^2$, and this foliation is Riemannian if we choose a suitable metric. For example, we choose the metric along $[0,1]$ to be standard and require each torus to be orthogonal to this direction. Then we define the vectors v and v' to be orthogonal in this metric and let the lengths of v and v' vary smoothly over $[0,1]$ so that $\|v\|(0) = \frac{3+\sqrt{5}}{2}\|v\|(1)$ and $\|v'\|(0) = \frac{3-\sqrt{5}}{2}\|v'\|(1)$. Let $\overline{v} = a(t)\,v$, $\overline{v'} = b(t)\,v'$ be the resulting renormalized vector fields. The basic manifold is a torus, and the isotropy groups are all trivial. We use coordinates $(t,x,y) \in [0,1] \times T^2$ to describe points of M. The basic forms are:

$$\Omega_b^0 = \{f(t)\}$$

$$\Omega_b^1 = \{g_1(t)\,dt + g_2(t)\overline{v}^*\}$$
$$\Omega_b^2 = \{h(t)dt \wedge \overline{v}^*\},$$

where all the functions are smooth. Note that $d\overline{v}^* = -\frac{a'(t)}{a(t)}\,dt \wedge \overline{v}^*$ *By computing the cohomology groups, we get* $h^0 = h^1 = 1$, $h^2 = 0$. *Thus, the basic Euler characteristic is zero.*

We now compute the basic Euler characteristic using our theorem. There is only one stratum, and the leaf closure space is S^1. *The foliation restricted to each leaf closure is an irrational flow on the torus. Thus,*

$$\begin{aligned}\chi(M,\mathcal{F}) &= \sum_i \chi\left(M(H_i)\diagup\overline{\mathcal{F}}\right)\chi\left(L_i,\mathcal{F},\ \mathcal{O}_{M(H_i)\diagup\overline{\mathcal{F}}}\right)\\ &= \chi\left(S^1\right)\chi\left(\{t\}\times T^2,\mathcal{F}\right)\\ &= 0\cdot 0 = 0,\end{aligned}$$

as we have already seen.

Following is an example of using the representation-valued basic index theorem, in this case applied to the Euler characteristic (Theorem 10.3).

Example 10.9. *Let* $M = \mathbb{R}\times_\phi T^2$ *be the suspension of the torus* $T^2 = \mathbb{R}^2\diagup\mathbb{Z}^2$, *constructed as follows. The action* $\phi\colon \mathbb{Z}\to \mathrm{Isom}\left(T^2\right)$ *is generated by a* $\frac{\pi}{2}$ *rotation. The Riemannian foliation* $\mathcal{F}$ *is given by the* $\mathbb{R}$*-parameter curves. Explicitly,* $k\in\mathbb{Z}$ *acts on* $\begin{pmatrix} y_1 \\ y_2\end{pmatrix}$ *by*

$$\phi(k)\begin{pmatrix} y_1 \\ y_2\end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}^k \begin{pmatrix} y_1 \\ y_2\end{pmatrix}.$$

Endow T^2 *with the standard flat metric. The basic harmonic forms have basis linebreak* $\{1, dy_1, dy_2, dy_1\wedge dy_2\}$. *Let* ρ_j *be the irreducible character defined by* $k\in\mathbb{Z}\mapsto e^{ikj\pi/2}$. *Then the basic de Rham operator* $(d+\delta_b)^{\rho_0}$ *on* $\mathbb{Z}$*-invariant basic forms has kernel* $\{c_0 + c_1 dy_1\wedge dy_2\colon c_0, c_1 \in \mathbb{C}$ *One also sees that* $\ker(d+\delta_b)^{\rho_1} = \mathrm{span}\{i dy_1 + dy_2\}$, $\ker(d+\delta_b)^{\rho_2} = \{0\}$, *and* $\ker(d+\delta_b)^{\rho_3} = \mathrm{span}\{-i dy_1 + dy_2\}$. *Then*

$$\chi^{\rho_0}(M,\mathcal{F}) = 2, \chi^{\rho_1}(M,\mathcal{F}) = \chi^{\rho_3}(M,\mathcal{F}) = -1, \chi^{\rho_2}(M,\mathcal{F}) = 0.$$

This illustrates the point that it is not possible to use the Atiyah-Singer integrand on the quotient of the principal stratum to compute even the invariant index alone. Indeed, the Atiyah-Singer integrand would be a constant times the Gauss curvature, which is identically zero. In these cases, the three singular points $a_1 = \begin{pmatrix} 0 \\ 0\end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ \frac{1}{2}\end{pmatrix}$, $a_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\end{pmatrix}$ *certainly contribute to the index. The quotient* $M\diagup\overline{\mathcal{F}}$ *is an orbifold homeomorphic to a sphere.*

We now compute the Euler characteristics $\chi^\rho(M,\mathcal{F})$ *using Theorem 10.3. The strata of the foliation are as follows. The leaves corresponding to* a_1 *and* a_3 *comprise the most singular stratum* M_s *with isotropy* $\mathbb{Z}_4$, *and the leaf correspondng to* a_2 *is its own stratum* M_l *with isotropy isomorphic to* $\mathbb{Z}_2$. *Then*

$$\chi\left(M_s\diagup\overline{\mathcal{F}}\right) = 2,$$
$$\chi\left(M_l\diagup\overline{\mathcal{F}}\right) = 1,$$

$$\chi\left(M_0\diagup\overline{\mathcal{F}}\right)=\chi\left(S^2\setminus\{\mathit{3\ points}\}\right)=-1.$$

In each stratum (M_0, M_l, or M_s), the representative leaf closure is a circle, a single leaf, and each stratum is transversally oriented. The Euler characteristic $\chi^\rho(L_j,\mathcal{F})$ is one if there exists a locally constant section of the line bundle associated to ρ over L_j, and otherwise it is zero. We see that

$$\chi^\rho\left(L_j,\mathcal{F},\mathcal{O}_{M_j\diagup\overline{\mathcal{F}}}\right)=\chi^\rho\left(L_j,\mathcal{F}\right)=\begin{cases}1 & \text{if } M_j=M_0 \text{ and } \rho=\rho_0,\rho_1,\rho_2, \text{ or } \rho_3\\ 1 & \text{if } M_j=M_s \text{ and } \rho=\rho_0\\ 1 & \text{if } M_j=M_l \text{ and } \rho=\rho_0 \text{ or } \rho_2\\ 0 & \text{otherwise}\end{cases}.$$

Then Theorem 10.3 implies

$$\chi^\rho(M)=(-1)\begin{cases}1 & \text{if } \rho=\rho_0,\rho_1,\rho_2, \text{ or } \rho_3\\ 0 & \text{otherwise}\end{cases}+(1)\begin{cases}1 & \text{if } \rho=\rho_0 \text{ or } \rho_2\\ 0 & \text{otherwise}\end{cases}+(2)\begin{cases}1 & \text{if } \rho=\rho_0\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}-1 & \text{if } M_j=M_0 \text{ and } \rho=\rho_1 \text{ or } \rho_3\\ 2 & \text{if } M_j=M_s \text{ and } \rho=\rho_0\\ 0 & \text{if } M_j=M_l \text{ and } \rho=\rho_2\\ 0 & \text{otherwise}\end{cases},$$

■

which agrees with the previous direct calculation.

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