Leonhard Euler in Berlin

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MATHEMATICAL LIFE

Leonhard Euler in Berlin

1. Introduction

Three hundred years ago, on 15 April 1707, Leonhard Euler was born in Basel. The year 2007 therefore became a veritable Euler year (even without being so-named by some international agency), and his achievements have been generally exposed, scrutinized, and celebrated all over the world. Euler's work in mathematics is very likely the most voluminous and the most influential ever, not only for the depth and the variety of his results but also for his influence on how mathematics is written and taught. As an indirect proof of this statement we consider the fact that a comprehensive scientific biography of Leonhard Euler is still missing, in spite of the enormous amount of work devoted to special aspects of his life and his production.

The remarkable stability of his living conditions makes it easy to give a rough sketch of Euler's biography. Even though he was apparently not a prodigy, his enormous talents showed early. The son of a protestant priest, he took up university studies in his home town Basel at the age of 13, enrolling in various faculties. Euler studied theology, philology, and history, only afterwards turning to his real favourites, the sciences and, above all, mathematics, finishing off even with some physiology. In 1727 he wrote his dissertation in physics, presenting a theory of sound on the basis of which he applied for an open physics professorship in Basel. Since this application was unsuccessful, the young Euler decided to follow an invitation to the St. Petersburg Academy in the same year, solicited by the two sons of Johann Bernoulli with whom he had studied and who were already there. Euler began to work in St. Petersburg in the medical department, gradually shifting his subject towards astronomy, geography, and mathematics, while improving his standing — and his salary — in the academy. Eventually he became a professor of mathematics, in 1737. Two important events from his private life should also be mentioned: in 1733 he married his Swiss compatriot Katharina Gsell, the daughter of a well-known painter, and in 1738 he lost his right eye, probably as the result of an infection.

After the death of Peter the Great, political circumstances in Russia had become more and more unstable, with increasingly unsafe living conditions and a real threat to the existence of the academy. Euler and his wife were very disquieted by this. Thus, in 1741 he accepted an invitation by Frederick II (Frederick the Great), the new king of Prussia, to come to Berlin and help create a new Royal Academy of Sciences. He arrived there on 25 July 1741, and he was to stay until 1766. In that year, having become more and more disappointed with the king's handling of the

AMS 2000 *Mathematics Subject Classification.* 01A70.

Figure 1. Euler at the age of 30, the only existing portrait showing him with full eyesight; after an original by Brucker.
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academy and of him personally, Euler accepted a second invitation to Russia, this time by Catherine II (Catherine the Great). He returned to St. Petersburg and lived there until his death on 18 September 1783.

We devote these pages to the Berlin period in Euler's life, trying to highlight not only — and even not so much — his scientific work but also his many other activities. We hope this will give the reader a more complete picture of Euler, and perhaps a better appreciation of how much we owe him.

Acknowledgements. The author wishes to thank Eberhard Knobloch, Wolfgang Knobloch, and Rüdiger Thiele for many helpful conversations. He is also indebted to the Geheime Staatsarchiv Preußischer Kulturbesitz for its generous support.

2. People

Even for a person with the mental powers of Euler, projects and activities were always connected with people. In fact, Euler was not shy or even introvert. On the contrary, he enjoyed good company for conversation, games, and music, and he loved to appreciate and to be appreciated. Hence, it is worth taking a closer look at the two people who were, without any doubt, the most important for Euler's everyday life in Berlin, even though he met them in person only rarely: Frederick II, the king, and Pierre-Louis Moreau de Maupertuis, the president of the academy.

Frederick II. Frederick II was certainly the most intellectually gifted monarch of his time, but he was also a very complicated character. All his life he felt attracted to philosophy, music, poetry, and above all, to French culture, while he disliked military exercises, and possibly the military as such. His father, Frederick William I,

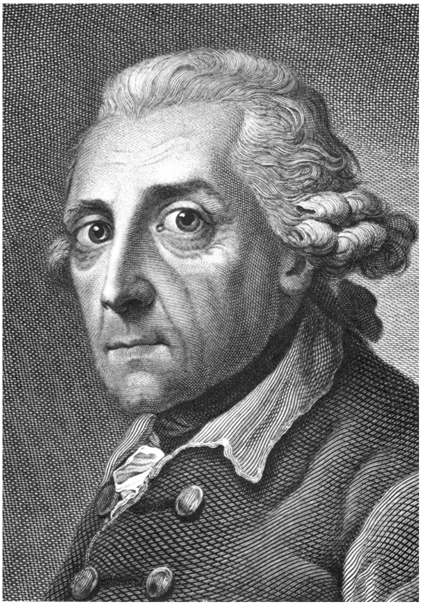


Figure 2. Frederick II, king of Prussia, contemporary engraving; Universitätsbibliothek Basel.
In: EO IV, 6, p. 276

had provided Prussia with much too large an army and a solid financial basis, at the expense of an authoritarian and parsimonious regime. Frederick suffered a lot in this atmosphere and tried to flee the country, together with his friend Hans Hermann von Katte. He was caught, his friend was executed, and he was kept as a prisoner by his father for two years. In 1740, at the age of 28, Frederick ascended to the throne and was suddenly confronted with the realities of power, without much preparation. In fact, he had spent the past years in the somewhat remote castle of Rheinsberg, with a circle of friends who shared his enthusiasm for French culture and the arts. Two decisions of his first weeks in power highlight the very different aims he now wanted to unite: first, he started an unjustified war for the rich province of Silesia belonging to the Habsburg Empire, and second, he tried to restore the *Société des Sciences*, the first precursor of the Prussian Academy of Sciences, which was founded by Leibniz but had deteriorated very much during the reign of Frederick William I. The latter decision is important for our context because it was what brought Euler to Berlin in 1741. The first decision was important, too, because it kept Frederick busy until the year 1763, the end of the Seven Years War — incidentally, the first war that was really global in the sense that it was fought on several continents. Necessarily, the extended warfare crucially limited the king's potential for attending to the arts and sciences, with the exception of a short period between 1745 and 1753. This is also the period during which Euler felt most happy being in Berlin, while after 1763 he felt more and more alienated and thought about going back to Russia, an idea which became reality in 1766.

Euler liked Petersburg and the people he met there very much, and he managed to keep very friendly relations with his colleagues in the academy and with the Russian government even after he had left for Berlin. That he left at all, after the invitation by Frederick to come to Berlin and reorganize the old *Société des Sciences* by attracting other prominent scientists and mathematicians to Berlin, is probably due to the political instability and the ensuing uncertainty in daily life after the death of Peter I, a time of unrest especially oppressive for Euler's wife Katharina. But once he had accepted the call to Berlin, he was enthusiastic about the possibilities for organizing a new academy which, according to the goal set by Frederick, should soon be among the leading institutions of this kind in the world. Euler was apparently under the impression that Frederick would entrust him with the whole architecture of this new academy and also with shaping its scientific, organizational, and administrative details. This, however, was not so: even before inviting Euler, Frederick had invited Pierre-Louis Moreau de Maupertuis to become president of the new academy. He promised to give the new president full powers for making decisions, limited only by the king himself, and Maupertuis had immediately accepted this offer, though he took office only in 1746. Even then and until his resignation in 1756, Maupertuis was not continuously present in Berlin, and his stays were interrupted by long periods of absence. Thus, Euler was in fact running the academy in all practical aspects, though formally he was only the director of the mathematical class. But he did not even become president when Maupertuis resigned, and this came as a bitter surprise to Leonhard Euler: he had apparently misjudged from the beginning his relationship with the king, regarding himself as the leading scientific counsellor. For the king, however, there was a clear difference between a philosophical approach to government — which he claimed for himself — and the influence of scientific advice and judgment on political decisions. Though Frederick was fully aware of the role of technology for the welfare of states and their political standing, he did not think scientists in general should speak out on political matters and discuss politics. He certainly appreciated their usefulness, but he wanted to keep them at a distance from the centre of decisions (he once wrote to Voltaire that the king should keep a scientific academy as a country squire keeps a pack of hounds, a statement of somewhat exaggerated poignancy that nevertheless seems to reveal a deep conviction not seldom found in politics even today).

The philosopher Voltaire, a leading protagonist of the French Enlightenment, had been in contact with Frederick already some time before his enthronement, at which occasion Voltaire rejoiced that now the sciences and the arts had come to power. He had visited Berlin already in 1740 and he stayed there continuously from 1750 to 1753, shaping to a large extent the style of conversation cultivated in Frederick's milieu. Leonhard Euler certainly did not fit this ideal of a courtier: his French was not brilliant, his speech was free of irony and cynical jokes, and his way of arguing showed the influence of Calvinist sermons and mathematical reasoning. Besides, unfair as it may seem, Frederick disliked his one-eyed appearance, calling him a 'cyclops' in private conversations. Euler certainly felt this fundamental difference but tried, in the tradition of the Enlightenment, to convince Frederick of his abilities and his efficiency by showing that mathematical thinking and its practical applications provide the best means to further the interests of the state and the government. It seems that he worked hard to reach this goal. For example, immediately after his arrival in Berlin he wrote an essay entitled "Commentatio de matheseos sublimioris utilitate",[1] which, however, was not published before 1847. A second example is provided by his book on artillery, on which we will comment later.

This approach conformed very well with Euler's firm Calvinist faith, since he believed that God reveals some of the secrets of the Creation through mathematics, allowing in this way for man-made improvements of the human condition. But in propagating this, Euler confronted the anti-religious tendencies of the French Enlightenment and thus of the Prussian court. Only when he finally understood that he would not be able to bridge this discrepancy, in spite of all the achievements harvested through his exceptional abilities, did he leave Prussia for St. Petersburg.

Pierre-Louis Moreau de Maupertuis. Pierre-Louis Moreau de Maupertuis was a celebrity when he was asked to become president of Frederick's academy. He had a solid mathematical education which he had acquired in England and in Basel with the Bernoullis. He was among the first scientists on the continent who adopted and popularized Newton's theories, and he got quickly involved in the attempts to clarify Newton's prediction about the flattening of the earth near the poles, quite contrary to what the Cartesians held to be true. After having been employed as a geometer by the French Academy in 1731, Maupertuis worked hard to secure funds for measuring a meridian in Lapland and thus being able to prove or disprove Newton's assertions. He was successful in 1736, and he returned from this strenuous expedition with convincing data which established Newton's victory over Descartes and made Maupertuis the most famous scientist of his time. But his health had suffered greatly from the strains of the expedition, another reason for his many absences from meetings of the academy and his frequent voyages to France, especially during the German winter, which was very hard on him.

Even though Maupertuis' career as a productive scientist had ended already in 1732, he began his presidency in Berlin with a spectacular result, namely, his

[1] "Commentary on the usefulness of higher mathematics", cf. E 790. Here and below we refer in this way to the numbers in the bibliography of G. Eneström, *Gustaf Eneström*: *Verzeichnis der Schriften Leonhard Eulers*, in: *Jahresbericht der Deutschen Mathematiker-Vereinigung* **4** (1910), Ergänzung 1. We will also refer to the collected works of Euler (EO — Euler's Œuvres), which have been appearing since 1911 in Basel, now with the publisher *Birkhäuser*, in four series and more than 70 volumes so far. For example, EO IV, 6 refers to volume six of the fourth series of the collected works.

Figure 3. Pierre-Louis Moreau de Maupertuis, engraving after the famous polar expedition of 1736; Universitätsbibliothek Basel.
In: EO IV, 6, p. 29

'principle of least action', published in 1746. In this work he asserted in vague terms, and without making any use of the new infinitesimal calculus, that the quantity of 'action' he had introduced was minimal in any dynamical transition of physical states. He based this very general assertion on metaphysical reasoning which should explain the efficiency, if not parsimony, of the Creator. Euler could have claimed priority in this matter, since he had published extensively on a general theory of curves that maximize or minimize certain functionals given by integrals, starting in 1736 and culminating in the book from 1744, entitled *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*,[2] which presented the first systematic treatment of the calculus of variations and was called by Constantin Carathéodory "the most beautiful mathematical book ever written." Quite obviously, Euler's mathematical treatment was far superior to Maupertuis' reasoning, and in particular, he emphasized from the beginning that it is not always a question of minimizing but that maxima can also occur in nature (and are described in the same way). However, he did not object to Maupertuis' assertion that his all-embracing principle of least action had been nicely illustrated by Euler's work. Euler even explained at some length that the principle of Maupertuis was not a mathematical but a philosophical statement and hence had no place in the world of mathematics. This statement was certainly compatible with Euler's basic convictions, but it also showed a certain respect for the powers that be. After all,

[2] *A method to invent curved lines with minimal or maximal properties*, cf. E 65.

Frederick had bestowed upon the president of the academy the full power to decide everything whatsoever, and Euler always respected this fundamental rule. Only this careful respect for the president's position made it possible for him to deal with each and every matter and to carry out in effect what Maupertuis ordered or might have ordered. The rather substantial — and extant — exchange of letters between Maupertuis and Euler, due to the frequent absence noted above of the president from the academy, shows that the two men were, if not outright friends, at least very smoothly cooperating partners who maintained a style of mutual understanding and respect. Maupertuis was a politician and a courtier, which Euler was not, but he was also a gifted and educated scientist who could understand what Euler thought and intended. The subtle balance in their relationship was quite vital for the very positive development of the academy in Berlin from 1741 to 1752.

Family and friends. Saying that Euler was definitely not a courtier does not at all mean that he was not a person who loved company and was not able to brilliantly entertain his guests. Nor does it say that Euler was unable to associate with or even form friendships with members of the higher society. On the contrary, his big house on what is today Behrenstraße not only had enough room for his large family but also regularly housed visitors from abroad. Euler was certainly a family man, who was reported to work at his desk amidst his many children without any sign of impatience.[3] Whatever he was doing he could interrupt at any time, taking up a completely different matter and then returning, seemingly uninterrupted, to his concentrated work.

Besides his nice house, Euler owned an orchard and a farm (which was recently identified by Wolfgang Knobloch; see Fig. 4), where his mother lived as a widow until her death.

Among the many friends of Euler was the Swiss medallist Johann Carl Hedlinger (1691–1771). The two compatriots had met in Russia, where Hedlinger had done some work for Empress Elisabeth and her court. He was then permanently employed by the Swedish king, and generally known as one of the most brilliant medallists in the world. His fame had also reached the Prussian court, and Frederick's counsellor in all matters of architecture and the arts, Georg Wenzelslaus von Knobelsdorff, had written to Hedlinger to inquire about what conditions might induce him to come to Berlin. But the letter remained unanswered, since Hedlinger was then in Switzerland, and Euler stepped in and invited him and his wife to his home for an extended visit. From all we know, Hedlinger's stay in Euler's house must have been very pleasant except for the fact that Frederick II did not show any interest at all in him, apparently angered by the negligence shown by Hedlinger in reaction to his generous offer. Only after Hedlinger had already spent six months in Berlin did Frederick write a letter to Euler saying that he would like to employ Hedlinger, either permanently or for a couple of years, at any salary Hedlinger might request. However, it was now too late: Hedlinger was already resolved to go back to Sweden, from where impatient signals kept asking for his return. But as

[3] Euler had 13 children with Katharina, of whom only 5 reached adulthood, and only his 3 sons Johann Albrecht (1734–1800), Karl Johann (1740–1790), and Christoph (1743–1808) survived him.

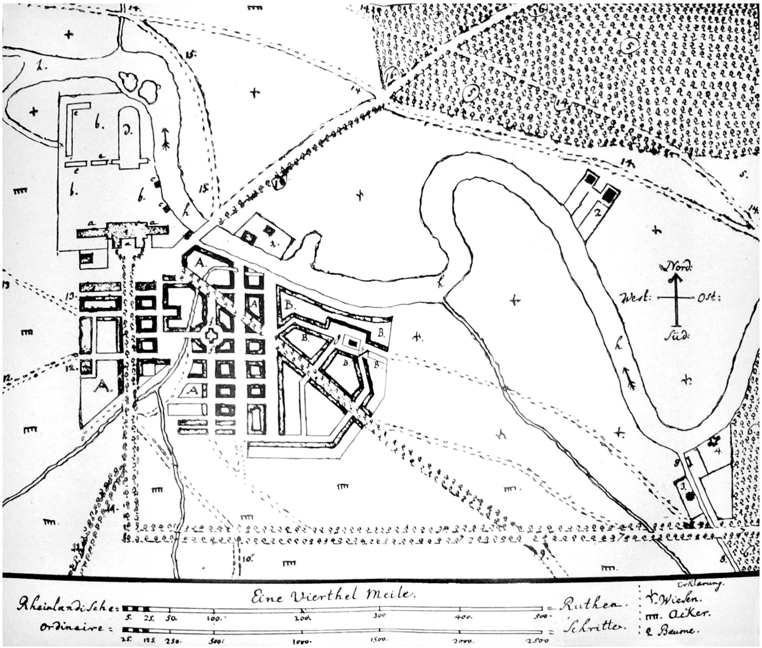


Figure 4. Euler's farm in Lietzow, now Charlottenburg, a part of Berlin; Euler's possessions are marked with the letter B.
In: W. Gundlach, *Geschichte der Stadt Charlottenburg*, 1905, Beilage XII

much as Hedlinger hated the style of conversation at the Prussian court, like his friend Euler he admired Frederick as the brilliant monarch of the Enlightenment, and he had long wanted to produce an image of him. Thus, during his stay in Berlin he had made preparations, though he finished the work only much later. In 1748 he produced a prize medal for the academy (see Fig. 5), based on discussions with both Euler and Maupertuis, who were so impressed that they elevated Hedlinger to the state of honourary member of the academy. In 1750 he completed his engraving of Frederick's image. The king was very delighted with this work and wanted to buy the stamp at any cost, as he wrote to Euler, but soon afterwards he no longer remembered that he wanted to pay anything to Hedlinger: in spite of all his talents and all his brilliance, Frederick II remained the son of his father in trying to save money wherever he could. Many projects fell short of expectations or were not implemented at all because of his parsimonious attitude.

3. Projects

The reform of the academy. As mentioned earlier, the academy in Berlin was founded by Gottfried Wilhelm Leibniz in the year 1700 under the name *Société des Sciences*. After a slow development in the first decade of its existence, it was finally opened officially on 19 January 1711. It resided in a building on the

Figure 5. Prize medal of the Société des Sciences et des Belles Lettres, Medailleur Johann Carl Hedlinger, Berlin 1747/48, ∅ 67 mm.
© Staatliche Museen zu Berlin-Preußischer Kulturbesitz, Münzkabinett

avenue *Unter den Linden* known as *Der neue Marstall*, where the older Royal Academy of the Arts had found its quarters in the year 1700. This 'cohabitation' of scientists and artists and their usually quite different institutions was at that time unique in Europe. The alimentation of the academy was effected through the so-called *Kalender-Privileg*, a monopoly on the production of calendars and related publications throughout Prussia. In this way a reliable though limited financial basis was provided which, however, did not allow the realization of the ambitious dreams of Leibniz. Moreover, the main burden connected with the bread-and-butter work of calendar-making fell upon the mathematicians and astronomers in the academy, a fact that regularly gave rise to some dissatisfaction. In addition, during the reign of Frederick William I (1713–1740) the *Société des Sciences* deteriorated due to the total neglect if not contempt from the side of the king, who installed his jester as its president. Nevertheless, some outstanding scholars were working in Berlin even then, like the philologist and botanist Johann Leonhard Frisch.

As we mentioned, Frederick II was prepared to change things as soon as he came to power, and to this end he sent out calls to some of the most prominent scientists of Europe, preferring, of course, those conforming with his ideal, the philosophical attitude of the French persuasion. When Frederick II invited Euler to Berlin, he probably did not know of Euler's enormous energy, extending to everything in his surroundings that offered possibilities for effective treatment. Thus, Euler immediately began to reorganize the academy in correspondence with the instructions by envoys of the king that his goal should be an institution rivaled only by the Paris and London academies. Euler's patience was certainly severely challenged by the first two Silesian wars, which absorbed all the energy of the government and left little room for matters concerning the academy. But Euler worked tirelessly, relying on his Swiss tenacity, and certainly not without effect. The long-interrupted series of academy publications came to new life, with the first volume, like many others to come, full of publications authored by Euler. Also, he insisted that meetings of the academy should be held regularly and with substantial protocols. Moreover, he brought new people to Berlin, and constantly developed his communication network, which included most of the significant scientists of his time. Fortunately, he was not alone: some influential people close to the king also wanted progress in academy matters, though their emphasis was more on the arts and literature. In 1743 a *Société de Belles Lettres* was founded as a further precursor of the promised new academy. Honouring these attempts, Euler worked on a unification of the two societies, and was successful on 24 January 1744, when the new *Académie Royale des Sciences et des Belles Lettres* was founded in the *Berliner Stadtschloss*.

The organization was entirely Euler's, from the conception of the statutes to the details of the opening ceremony, and it showed some remarkable aspects. Notably, among the four classes representing the sciences and the humanities, the *Classe de Philosophies* was unique in Europe, attesting to the important role this academy would soon play in the philosophical disputes and quarrels of the 18th century. As we can see from Fig. 6, Euler appears already as director of the mathematical

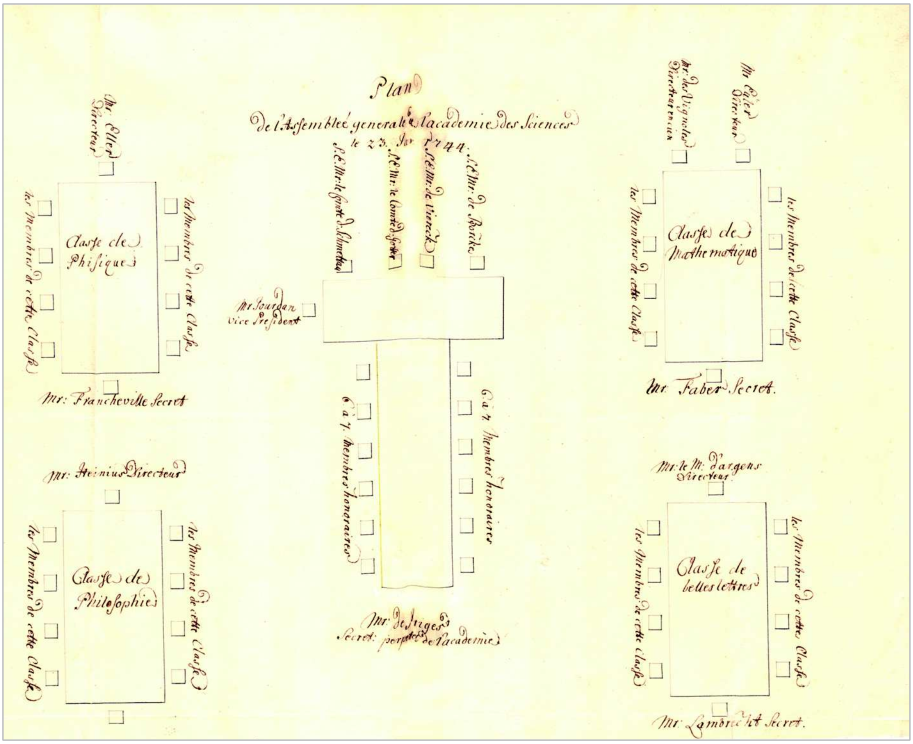


Figure 6. Sketch of the seating arrangement for the opening ceremony of the new academy, 24 January 1744.

class, in which function he was officially installed only on 3 March 1746. Thus, in spite of all the shortcomings and frustrations, Euler quickly built a sound basis for what was to become probably the most fruitful period of his life. In 1752, after

Figure 7. The academy building on *Unter den Linden*, its location from 1752 to 1904; drawing by Calau, engraving by Lauréns and Thiele. © Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften, Grafiksammlung, P/BON–1135

a fire had destroyed *Der neue Marstall*, both academies moved into a new and quite representative building, which had to make way for the new state library only in 1903 (Fig. 7).

Practical mathematics. As we have noted already, Euler was resolved to prove the power of pure mathematics through practical applications, not only because he wanted to convince his king of the possibilities of his own field, and hence of his own abilities, but also as a matter of principle, since it was his deepest conviction that it is through mathematics that we can get insight into the work and even the intentions of the Creator. Naturally, the academy had to serve the king and the state in many respects, in engineering projects, in questions of time measurement and calendar calculation, and regularly in the evaluation of technological innovations, acting very much like a modern patent office. Euler was involved in these activities a lot, including some greater projects. The first among these was the rebuilding of the Finow canal, a waterway which provided a direct connection between Berlin and the Oder river but had deteriorated over the years. Euler was a member of a commission which had to visit the canal, his task being the exact measurement of the level, on which corrections had to be based. Fortunately, not all his missions expanded into such strenuous expeditions.

Another well-known story concerns the fountain of Sanssouci and the dream of the king to have one rising to tremendous height such as to impress any visitor. It is usually told that Euler provided detailed recommendations which, however, did not correspond to reality, and that in fact Frederick never saw a fountain higher than 30 centimeters. This outcome in turn strengthened Frederick's contempt for mathematics and mathematicians. The latter statement is true, but some of the facts have to be corrected, as was shown by M. Eckert,[4] whose detailed account shows that Euler's analysis was quite appropriate and that his calculations would have given the desired result if only the parsimony of Frederick had not interfered again. For example, he twice selected completely inexperienced but cheap craftsmen and ordered them to use wood for the water-conducting pipes, a material which simply could not withstand the necessary pressure, though Euler had repeatedly insisted upon pipes made of lead.

Other tasks of Euler were closer to mathematics, for example, when the king inquired about the correct setup of a lottery in order to fill the notoriously empty coffers of the state. The first such lottery had been established in Berlin in 1740 but unfortunately did not live up to expectations. Euler was also asked to think about the basis for a life insurance system and a (restricted) pension system, ideas which attest to the modernity of Frederick's thinking.

We will now turn to a project with practical implications which was closer to Euler's mathematical thinking than those mentioned above, and which seems to have played a special role in his attempts to convince his king of the usefulness of mathematics. Finding the Prussian army involved in extended warfare, Euler must have thought hard about an application of mathematics in this context, especially one that would improve the performance of the Prussian weapons in an indisputable manner.

Knowing very well that successful applications on a larger scale are impossible without calibrating experiments, it must have come like a heavenly gift to Euler when he discovered the book *New principles of gunnery*, published in London in 1742. The author, Benjamin Robins, was not unknown to Euler, since he owed Robins a very negative criticism of his first book on mechanics written in St. Petersburg. However, as in the case of Maupertuis' priority, Euler cared very little about such matters and did not retaliate in his treatment of Robins' book, even though he found many mistakes which had to be corrected. In this way, he improved the work greatly and elevated Robins to a fame which he would never have achieved otherwise. At any rate, what Euler was interested in were the results of the experiments in which Robins had found a rather ingenious way to measure the true velocity of the cannon balls and the force of the gunpowder. With these data Euler was able to apply the infinitesimal calculus, thus creating the foundations of modern ballistics. Much to his satisfaction, he was able to improve tremendously the results and predictions of Robins, along the way correcting by a factor of two a formula given by Newton for the air resistance. The new book by Euler finally consisted in a translation of Robins work, enriched with extensive comments, theoretical developments, and calculations which resulted in a volume five times the size of the original, with the appropriate baroque title *Neue Grundsätze der Artillerie enthaltend die Bestimmung der Gewalt des Pulvers nebst einer Untersuchung über den Unterscheid des Wiederstands der Luft in schnellen und langsamen Bewegungen aus dem Englischen des Herrn Benjamin Robins übersetzt und mit den nöthigen*

[4] M. Eckert, "Euler and the fountains of Sanssouci", *Archive for the History of the Exact Sciences* **56** (2002), 451–468.

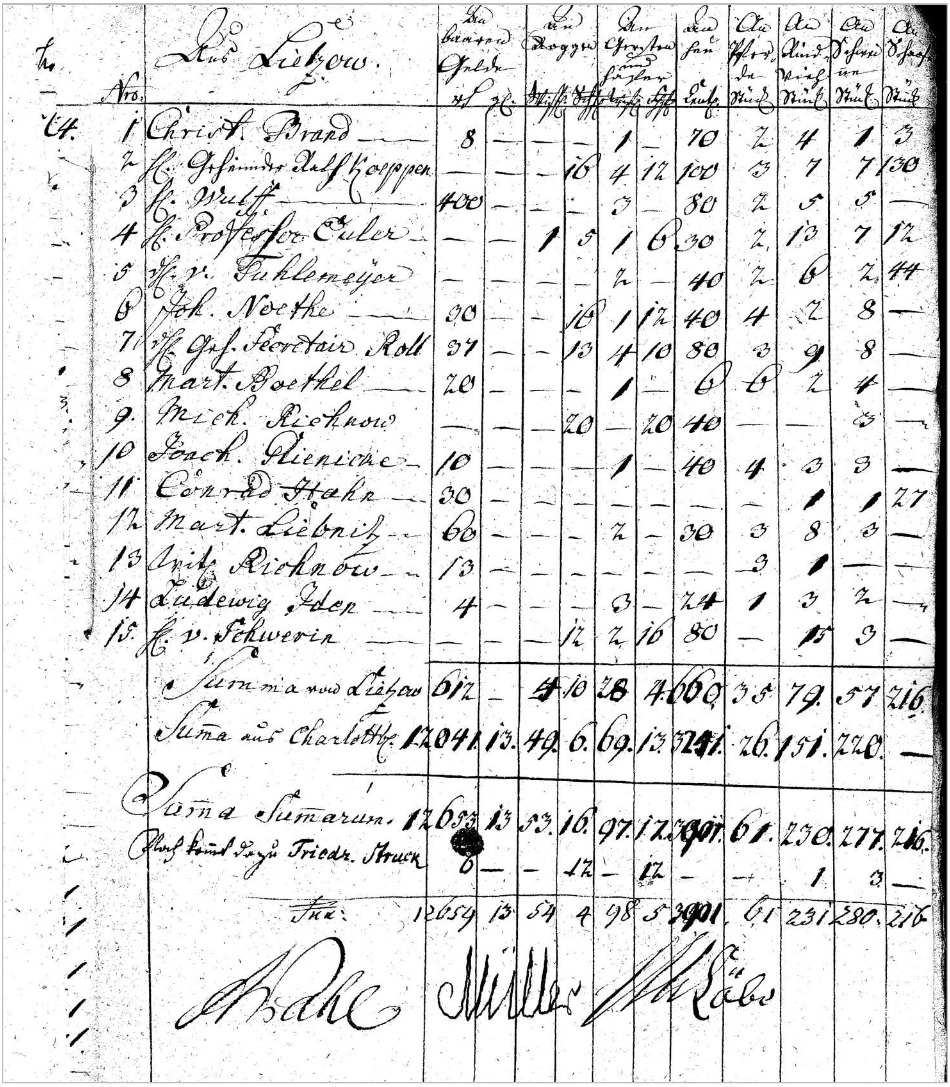


Figure 8. Official list of the damages caused by plundering in the village Lietzow in 1760.

Erläuterungen und vielen Anmerkungen versehen von Leonhard Euler königl. Professor in Berlin. Berlin bey A. Haude königl. und der Academie der Wissenschaften privil. Buchhändler. 1745.[5]

[5] *New principles of gunnery with determination of the true force of the gunpowder and an investigation of the different air resistance for fast and slow motions. From the English of Mr.*

Euler certainly met the success he intended, at least outside Prussia: his work was immediately introduced in military schools, notably in France, from where he received significant praise and a sizable remuneration; Napoleon, whose love for mathematics is well known, had to study Euler's book as a young officer. However, we do not have any proof of a similar reaction from the Prussian court. We know of a letter which Euler sent to Frederick II some time in 1744, wherein he announced his plan to translate and extend Robins' book and asked for permission to devote his time to this project, but we do not know of any answer. We also know of the letter of dedication to the king accompanying the completed book. This letter is dated 20 April 1745, and seems to be hitherto unpublished. Euler makes it quite clear, in spite of the formal modesty of his writing, that he regards this piece of work as the desired proof of the all-embracing power of higher mathematics. He describes the relationship between what he calls here 'theoretical mathematics' and its applications, which the king seems to have asked for, as the addition of experimental data, to determine the constants of integration, to the equations of motion which arise from general principles like his calculus of variations. It seems that the idea of equivalent but different theories describing the same phenomena was still alien to Euler. His religious convictions led him to believe in 'true equations'. Alas, we do not know of any comments or signs of favour from the side of Frederick in reaction to this remarkable achievement of Euler.

Private matters. As would be expected, Euler handled also his family business with great care and great efficiency. Thus, the Euler family could be called well-off, since Leonhard not only enjoyed a rather exceptional salary but also earned considerable money from the many academy prizes he won, among which were at least twelve prizes from the French Academy of Sciences. Besides that, he had revenues which must also have been significant from his farmland. At least we know that during the Russian–Saxon occupation of 1760 Euler lost, by the official record, 2 horses, 13 cattle, 7 pigs, and 12 sheep (see Fig. 8). For this damage he was reimbursed by the occupation forces and received, on top of this, a very generous compensation from Catherine the Great. This Russian generosity had, unfortunately, no parallel in Prussia, which eventually made it even easier for Euler to go back to St. Petersburg. There were probably many other sources of occasional monetary gains, a lottery prize for example, and a careful look at the famous Handmann portrait (Fig. 9) reveals that Euler is clad in silk produced from the Berlin academy's own mulberry plantation.

4. Scientific work

As mentioned above, a comprehensive treatment of Euler's scientific works is missing in the literature, despite an abundance of detailed discussions of specific aspects. In the framework of a single essay any in-depth study is ruled out if one tries to describe some extended period of Euler's life, as we do here. Nevertheless, it is easy enough to collect some statistics about what he was doing during his 25 years in Berlin. Thus, we can say that he prepared roughly 380 articles or books

Benjamin Robins translated and where necessary commented on and amended by Leonhard Euler, and so on.

Figure 9. Emanuel Handmann, Portrait of the mathematician Leonhard Euler; pastel on paper, 57 × 44 cm, 1753.

in this time, of which 275 were also published in the same period.[6] For this, we rely on the very careful catalogue of Euler's writings compiled by Eneström,[1] who comes to a total of 866 (counting also letters, prefaces, and the like but not reprints or translations). There is no way of briefly surveying this massive production, but the printed books provide a fairly accurate guide to the areas of interest to which Euler devoted a substantial portion of his time in Berlin. A complete list is given in Fig. 10. From the rich 'menu' that Euler has served us here, we want to select two important topics for somewhat closer scrutiny.

The birth of analysis. Euler's continuous and broad flow of work developed from various sources, for instance, from problems which had become famous because they had withstood the solution attempts of many illustrious colleagues, like the 'Basel problem'. Or it derived from personal interests like shipbuilding (it seems to be unknown why Euler, a Swiss citizen, was so fascinated with this), or from questions on optics, certainly fostered by his loss of eyesight, or from accidental impulses, like his work on the foundations of ballistics. Besides, Leonhard Euler was very familiar with the work of the giants of the past and eager to develop it further or even to surpass it by corrections or proofs of outstanding conjectures.[7] More importantly, in spite of being constantly absorbed by considerations directed at the solution of specific problems, it seems that he always kept in mind the theoretical

[6]The publication of Euler's scientific articles was not completed until 1862 (with Eneström's number E 856), almost 80 years after his death.

[7]That we talk about 'Fermat's Last Theorem' today is due to the fact that Euler had proved, one by one, all the other conjectures left behind by Fermat.

1744A	*Methodus inveniendi* Euler's groundbreaking foundation of the calculus of variations; E 65
1744B	*Theoria motuum planetarum et cometarum* Improved method for calculating the orbits of planets and comets; E 66
1745	*Neue Grundsätze der Artillerie* Translated and enormously enlarged edition of *New Principles of Gunnery* by B. Robins; E 77
(not before) 1745	*Anleitung zur Natur-Lehre* *Introduction to the Natural Sciences*; published posthumous in *Opera posthuma* II, 1862; E 842
1746	*Gedancken von den Elementen der Cörper* *Thoughts about the Elements of Bodies*; gives objections to Leibniz' theory of monads based on arguments from physics and theology; E 81
1747	*Rettung der göttlichen Offenbahrung gegen die Einwürfe der Freygeister* [anonymous] *Rescue of Divine Revelation from the Objections of the Freethinkers*; argues against atheistic tendencies of the enlightenment; E 92
1748	*Introductio in analysin infinitorum* *Elements of Analysis*; first part of the Analytic Trilogy; E 101,102
1749	*Scientia navalis* Encyclopedic work on shipbuilding and navigation, prepared in St. Petersburg 1738; E 110-111
1753	*Theoria motus lunae* So called first lunar theory; E 187
1755	*Institutiones calculi differentialis* *Introduction to the Differential Calculus*, prepared around 1748; E 212
1765A	*Theoria motus corporum solidorum* His foundational work on the mechanics of solids; E 289
1765B	*Théorie générale de la dioptrique* *General Theory of Lenses*, prepared in Berlin, published in 1862; E 844
1768A	*Lettres à une princesse d'Allemagne* Popular letters on science and philosophy, prepared in Berlin, but published in St. Petersburg, with unusual success: at least 12 French, 9 English, 7 German, and 4 Russian editions; E 343, 344, 417
1768B	*Institutiones calculi integralis* *Introduction to the Integral Calculus*; Euler's exhaustive book on the integral calculus was written in 1763 and appeared 1768-1770; E 342, 366, 385, 660

Figure 10. List of books which Leonhard Euler wrote or published while he was in Berlin, 1741–1766

framework he was working in, and that he was able to adapt the whole architecture of the relevant theory according to the new notions and arguments that arose in the process of his studies. Whenever he felt he had reached a certain level of maturity in his investigations, he put his ideas together in a book, and these books usually remained highly influential for a long time.

The process of development did not stop once he had written a book, but the improvements or modifications he introduced later in other publications often did not reach the public for quite some time: his educational style of writing was simply too convincing.

We shall try to exemplify this sketch of Euler's working habits with his 'analytic trilogy' consisting of the books marked 1748, 1755, and 1768 B in the list in Fig. 10. It is a well-known fact that Euler built the basis for what we today call 'analysis', that is, the theory of real functions, the various limit processes that lead to them, and the differential and integral calculus from which most of the interesting functions can be derived. What is maybe less well known is the fact that in Euler's work analysis arises for the first time on a foundation which is independent of geometry (while elementary algebra is presupposed and used as an appropriate substitute). It fits this picture that Euler, like most other early analysts, was called a 'geometer', but an even more striking illustration is provided by the total absence of diagrams, which had accompanied mathematical texts since the days of Euclid, and a closer look reveals that this effect is by no means compensated by an abundance of formulae: in Euler's texts the written language dominates by far.

If we look at the table of contents of *Introductio in analysin infinitorum*, we see that the book is divided into two parts, the first one establishing the foundations of analysis, the second one building what we would today call linear algebra. That the two parts arise in this order is another proof of the above statement that, with Euler, analysis arises independently of geometry. The main goal of the first part is the study of functions with real arguments and real values, though occasionally complex numbers are also considered. Euler does not give a motivation for this, since he could assume that his readers already knew enough examples for the overwhelming importance of functions for all applications of mathematics, and had some experience with polynomials and rational functions. He insists, however, (in § 4) that a function must be given by analytic expressions under which notion he comprises the basic algebraic formulae and the specific limit expressions which are the main topic of the whole book. Then he proceeds to introduce these limit expressions, notably infinite series, power series in particular, and infinite products. The last chapter is devoted to continued fractions.

He then makes systematic use of these operations to introduce the elementary transcendental functions, and as a particular highlight he presents the trigonometric functions as power series, again without any recourse to their geometric definitions. Along the way he shows what powerful conclusions can be reached by relating infinite series and infinite product representations of a function to each other. Carrying the decomposition of polynomials into linear factors over to entire functions, he solves the 'Basel problem' mentioned earlier, which asked for the precise value of the sum of the inverse squares of the positive integers; in fact, he computes the value of Riemann's (perhaps more correctly: Euler's) zeta function at all positive even integers.

It is fair to say that here Euler anticipates the modern (physics) concept of a partition function. As is well known, his arguments are not entirely satisfactory with respect to rigour, and some of his conclusions are indeed extremely bold. It seems, however, that the unfolding of Euler's thinking proceeded somewhat differently from what we take for granted today, a shift probably caused by the influence of Carl Friedrich Gauss. Let us take as an example the introduction of the exponential series in §§ 115 and 116 of the *Introductio*, which certainly leaves out all the necessary limit arguments but gives a very good motivation for the *form of the series*, which could hardly be guessed otherwise. Only much later would Euler complete this motivation to a proof with the remark that the exponential series satisfies the functional equation and hence interpolates rational powers; most modern textbooks (and lectures) give exact proofs but no motivation at all.

Thus, it seems that Euler was completely sure of the truth of his assertions but did not want to spoil the (convincing) flow of the exposition by leaving out a beautiful result. In addition, he was probably too impatient to postpone publication to a later date when he would have arrived at more satisfying proofs. And this he did in many cases, for example, in the case of the 'Basel problem', for which he provided many more proofs later, and for which he had numerical corroborations when he first published it.

The second volume in the trilogy concerns the differential calculus (volume 1755 in the list in Fig. 10) and builds explicitly on the *Introductio* just discussed. The preface is a marvelous piece to read (especially in Latin) and offers some striking features. First of all, Euler develops right away a very general notion of function which could be easily identified with our modern view, freeing the definition completely from any requirement of analytic expressions. This change has been little noticed, as can be seen from the fact that the modern definition of function is usually attributed to Dirichlet and thus placed a hundred years later. Next, Euler deals at length with the problem of the 'evanescent quantities', to make it perfectly clear that a quotient of quantities tending to zero can have a finite limit. He also indicates here that the task of determining these limits is of vital importance for applications, since the real problems posed by nature can only be understood by solving differential equations. These equations are derived, constituted, and solved by infinitesimal procedures only; we would not be surprised if Euler had added here that "nature is simple only infinitesimally", a remark due to Einstein.

A further interesting feature of this preface consists in the fact that Euler illustrates the reasoning of this introduction by just one example, namely the firing of cannon balls (which he had dealt with at length, as we know). Moreover, he calls the differential quotient in this connection the *ratio ultima* which must have had an ironic connotation for his contemporaries who knew that Frederick II had written *Ultima Ratio Regis* [*the last resort of the king*] on his cannons, following the example of Richelieu. This nice pun may explain why Euler did not illustrate the goals of the calculus by explaining Newton's brilliant derivation of Kepler's laws.

That the ensuing chapters present the material of higher differential analysis very much in the way it is presented in most calculus courses nowadays may come as a surprise for some, but is easily explained by the fact that Euler's work has been copied ever after, at least indirectly, notwithstanding the many refinements and extensions. This is also true of the third volume in the analytic trilogy, which is devoted to the art of integration and will always remain its true and lasting foundation.

Another striking example of Euler's continuously developing thinking and the danger of overlooking an important result of his is provided by the theory of the zeta function.[8] It has been known for some time that Euler not only derived the Euler product of zeta but also proved the functional equation, or more precisely, that E. Landau turned Euler's arguments into a valid proof — the way Euler guessed this relation is truly admirable! What is apparently not mentioned in the literature is that Euler also wrote down the Mellin transform which was Riemann's basis for his analysis of zeta; this can be found in *Novi comment. Acad. Sc. Petrop.* **14** (1769), 129–167 (this information we owe to Nobushige Kurokawa).

Euler's architecture of analysis convinces by its methodological consistency and not so much by its rigour, to which he did add, though, in later work. It furnishes a very impressive proof of Euler's analytic instinct that his edifice lived through many logical crises only to become reinforced and augmented, without significantly changing its structure or losing its beauty.

Educational writings. In the thinking of Leonhard Euler we find a remarkable emphasis on the presentation of the many insights he had. He always tried to make his thinking clear to other people, and not only to those of comparable insight, as if the truth of a thought could only be established by communicating it successfully; here again it seems that we meet the influence of the Calvinist sermon. Be this as it may, Euler engaged himself in mathematical education early on and this occupation accompanied him all his life. Already in 1735 he wrote a very successful schoolbook on arithmetic (E 17: *Einleitung zur Rechen-Kunst, zum Gebrauch des Gymnasii bey der Kayserlichen Academie der Wissenschafften in St. Petersburg. Gedruckt in der Academischen Buchdruckerey* 1738; the second part appeared in 1740).

In Berlin he conceived his educational masterpiece, a collection of 234 letters to a German princess on questions of physics and philosophy. These letters were written to the daughter of the margrave of Brandenburg-Schwedt with whom Euler kept very friendly relations. Thus, he was asked to give private lessons to the margrave's daughter, then 16 years old, in science and philosophy. When, in the course of the Seven Years War, the Prussian court left Berlin temporarily, anticipating the short occupation of Berlin by Russian and Saxon troops in 1760, Euler was forced to continue his lessons by letter. Apparently, he was resolved already then to combine these letters into a book, which nevertheless appeared only much later, when he was already back in St. Petersburg (E 343, 344, and 417: *Lettres à une princesse d'Allemagne sur divers sujets de physique et de philosophie Tome premier A Saint Pétersbourg de l'imprimerie de l'académie impériale des sciences* 1768; the second part appeared in 1769, and the third in 1770). Reading this book is a pleasure even today, and one cannot but admire the clarity and the ease with which Euler explains such difficult matters as the constitution of the solar system according to Newton or the six possibilities for measuring longitude at sea which were in use or under examination in Euler's days. (We note in passing

[8]See R. Ayoub, "Euler and the zeta function", *Amer. Math. Monthly* **81** (1974), 1067–1086.

that Euler contributed substantially to the eventual solution of this problem and received a (small) gratification from the English parliament, which had offered a sum up to £ 20,000 for any method which would make it possible to measure longitude with sufficient precision; larger portions of this money were allotted to the clockmaker John Harrison and to the mathematician and geographer Tobias Mayer, who in turn had made substantial use of Euler's theory in compiling his lunar tables.) Other remarkable parts of Euler's letters concern an exposition of elementary logic, where we encounter what is now known as Venn diagrams, Euler's own theory of light and sound which to some extent predated the optical theories of the 19th century, and his review of the major philosophical positions of the 18th century and their origins. That Euler was understandable when writing about such complicated matters is best proved by the fact that this collection of letters was among the most economically successful books of the 18th century and has gone through many reprints and translations in the almost 250 years which have passed since its first appearance. Even a modern reader will enjoy Euler's remarkable clarity of exposition, although by necessity some parts have become obsolete with time.

Another project of pedagogical as well as scientific nature concerned an atlas designed for schoolchildren in Prussia. This enterprise, initiated by the king and ordered by the president of the academy, showed Euler in all his capacities in an exemplary way. As we would expect, Maupertuis entrusted Euler with the details of the project, which comprised the selection, construction, and design of the maps to be shown, the selection, contracting, and paying of the craftsmen who should work on the project, and finally the printing and the distribution of the completed work. Scientifically, the resulting book showed a few innovations, such as unusual projections or a global map of the lines of magnetic aberration. For the practical use of schoolchildren a much smaller format was chosen, differing greatly from the usual atlas formats, and this example quickly set new standards. The book appeared in 1753, followed by a second edition in 1760.

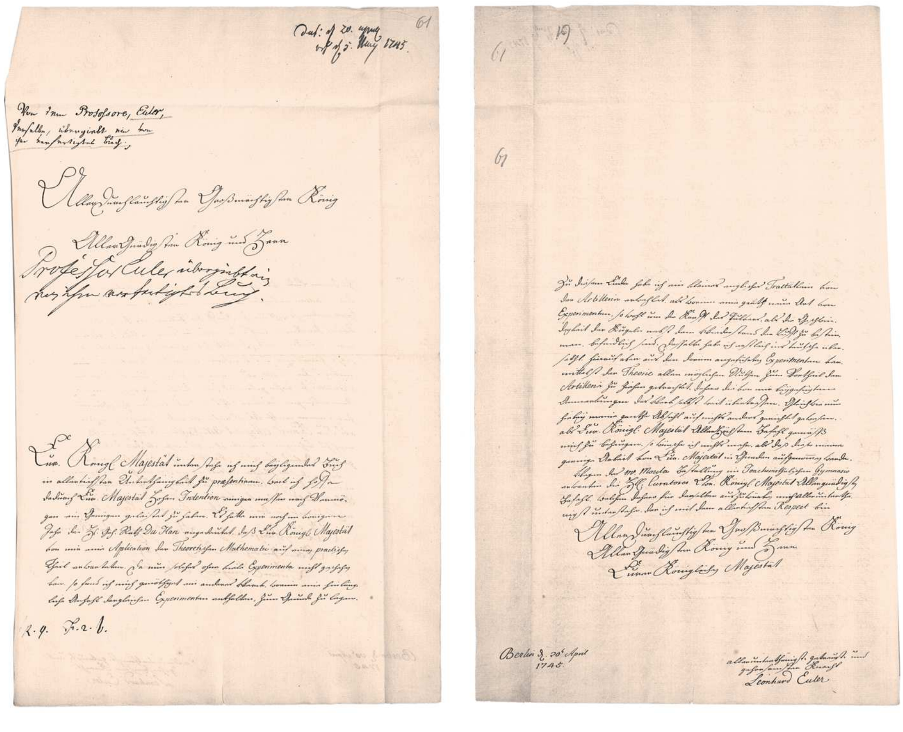
The final masterpiece in Euler's educational work should be the *Vollständige Anleitung zur Algebra von Hrn. Leonhard Euler.* 1. *Theil. Von den verschiedenen Rechnungs-Arten, Verhältnissen und Proportionen. St. Petersburg, gedruckt bey der Kays. Acad. der Wissenschaften* 1770; the second part appeared in the same year (E 387, 388). Euler's pedagogical fame is underlined by the well-known story (or perhaps legend) that the servant to whom Euler, then completely blind, dictated the manuscript in St. Petersburg was afterwards quite competent in algebra, although he had never received a formal education. Though the book was published in St. Petersburg, there is little doubt that the main body of the material had been outlined already in Berlin.

It certainly underscored Euler's qualities as a mathematical educator when Gauss said: "The study of all of Euler's papers will remain the best and irreplaceable school for all areas of mathematics."

All things considered, Leonhard Euler was an exceptional human being and a singular scientist and mathematician, to be compared only to Archimedes, Newton, and Gauss. His habit of tireless work, carried out with the greatest care and indefatigable energy, was based on the firm belief that the world can be understood and changed for the better by applying the scientific method of rational explanation. He applied his seemingly unlimited intellectual powers to all questions he was confronted with, always looking for new concepts and fruitful ideas, even before he had a definite method to handle them. His only major limitation arose from his religious faith, which was at the same time the major source of his strength, deeply rooted in his Swiss origin and his Calvinist family background.

Appendix

In this appendix we give the full German text of Euler's dedicatory letter to Frederick II which accompanied the first copy of his book *Neue Grundzüge der Artillerie*; this letter is not printed in EO and is perhaps not known so far. We then give a translation into modern English which does not capture the specific style of the time but hopefully the main content; we restrict the translation to the main part of the letter concerning the artillery book. The author thanks Wolfgang Knobloch for help with reading Euler's handwriting.


Berlin d. 20. April 1745.

Leonhard Euler

Figure 11. Dedication letter from Leonhard Euler to King Frederick II, accompanying his book on artillery.

Text of the letter.
[Von dem Professore Euler
derselbe übergiebt ein von
ihm verfertigtes Buch]

AllerDurchlauchtigster Großmächtigster König
AllerGnädigster König und Herr

[Professor Euler übergiebt ein
von ihm verfertigtes Buch]

Eurer Königlichen Majestät unterstehe ich mich beyliegendes Buch in allertiefster Unterthänigkeit zu praesentiren, weil ich hoffe dadurch Eurer Majestät Hohen Intention einiger massen nach Vermögen ein Genügen geleistet zu haben. Es hatte mir noch im vorigen Jahr der Herr Geh. Rath Du Han angedeutet daß Eure Königliche Majestät von mir eine Application der Theoretischen Mathematic auf einen practischen Theil erwarteten. Da nun solches ohne viele Experimenta nicht geschehen kan, so fand ich mich genöthiget, ein anderes Werck, worinn eine hinlängliche Anzahl dergleichen Experimenten enthalten, zum Grunde zu legen.

Zu diesem Ende habe ich ein kleines englisches Tractätlein von der Artillerie erwehlet, als worinn eine gantz neue Art von Experimenten, so wohl um die Krafft des Pulvers, als die Geschwindigkeit der Kugeln nebst dem Wiederstand der Lufft zu bestimmen, befindlich sind. Dasselbe habe ich erstlich ins teutsche übersetzt, hierauf aber aus den darinn angeführten Experimenten vermittelst der Theorie allen möglichen Nutzen zum Vortheil der Artillerie zu ziehen getrachtet. Dahero die von mir beygefügten Anmerkungen das Werk selbst weit übertreffen. Gleichwie nun hirbey meine gantze Absicht auf nichts anders gerichtet gewesen, als Eurer Königlichen Majestät AllerHöchstem Befehl gemäss mich zu bezeugen, so wünsche ich nichts mehr, als daß diese meine geringe Arbeit von Eurer Majestät in Gnaden aufgenommen werde.

Wegen des Mr Moula Bestallung im Joachimsthalischen Gymnasio erwarten die Herren Curatores Euer Königlichen Majestät Allergnädigsten Befehl, welchen dahero für denselben auszuwürken mich allerunterthänigst unterstehe, der ich mit dem allertiefsten Respect bin

AllerDurchlauchtigster Großmächtigster König
Allergnädigster König und Herr
Eurer Königlichen Majestät

Berlin d. 20^{ten} April
1745

allerunterthänigst ~ getreuest ~ und
gehorsamster Knecht

Leonhard Euler

Translation. [The letter begins with notes from the registrar and a verbose salutation formula, which we leave out.]

To Your Majesty I dare present most humbly the enclosed book, since I hope to have contributed in this way towards Your Majesty's high intentions, according to my abilities. Privy Councillor Du Han had indicated to me last year that Your

Majesty expects from my side an application of Theoretical Mathematics to some practical matter. Since such applications are not possible without many experiments, I saw the necessity to base my work on a book which already contained sufficient experimental data.

For this purpose I selected a small English treatise on artillery which describes an entirely new kind of experiments to determine the force of the gunpowder as well as the velocity of the cannon ball with the air resistance considered. First I translated the book into German, and then I tried to derive as much benefit as possible for the practice of artillery from using those experimental data in the theory. Thus, the comments I added are more voluminous than the original book itself. As it was my only intention all the time to show my compliance with Your commands, it is now my only wish that this little work of mine would be gracefully received by Your Majesty.

[The remaining lines of the letter concern a Swiss compatriot for whom Euler wanted to get, and got, a position in Berlin.]

J. Brüning