Fractal analysis for sets of non-differentiability of Minkowski's question mark function

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Abstract: We consider Minkowski's question mark function Q on the unit interval $\mathcal{U}$. We show that $\mathcal{U}$ can be written as the union of the three sets

$$\Lambda_0 := \{x : Q'(x) = 0\}$$

$$\Lambda_\infty := \{x : Q'(x) = \infty\}$$

$$\Lambda_\sim := \{x : Q'(x) \text{ does not exist and } Q'(x) \neq \infty\}.$$

The main result is that the Hausdorff dimensions of these sets are related as follows.

$$\begin{aligned} & \dim_H(\nu_F) \\ < \ & \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H\left(\mathcal{L}(h_{\mathrm{top}})\right) \\ < \ & \dim_H(\Lambda_0) = 1. \end{aligned}$$

Here, $\mathcal{L}(h_{\mathrm{top}})$ refers to the level set of the Stern-Brocot multifractal decomposition at the topological entropy $h_{\mathrm{top}} = \log 2$ of the Farey map F, and $\dim_H(\nu_F)$ denotes the Hausdorff dimension of the measure of maximal entropy of the dynamical system associated with F.