Fefferman Constructions in Conformal Holonomy

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Chapter 1

Introduction

R. W. Sharpe's book [56] explains that differential geometry is, ultimately, the study of a connection on a principal bundle. A beautiful realization of this idea are Cartan's generalized space, which generalize both semi-Riemannian and homogeneous geometries. Heuristically, a Cartan geometry of type (G, P) – where P is a closed subgroup of a Lie group G – is a curved version of the homogeneous space G/P. This idea is encoded via a special kind of connection one-form taking values in the Lie algebra of G, which is defined on a principal bundle with structure group P (cf. Chapter 2.1).

Among the very rich class of Cartan geometries, parabolic geometries provide in many ways the best confirmation that differential geometry is the study of a connection on a principal bundle. As is detailed in Chapter 2.3, these Cartan geometries induce underlying geometries on the base manifolds and there always exists a unique, canonical connection for basically all such underlying geometries. Because of this fundamental property and other powerful machinery flowing from it, and because the underlying geometric structures in this class are quite general and include a number of important examples from differential geometry, there has been a lot of interest in parabolic geometries in the recent literature. (For a survey of some of the main advances, see A. Čap's survey paper [14].)

Perhaps one of the most important examples of a parabolic geometry is provided by semi-Riemannian conformal geometry (cf. Chapter 3.1; two other interesting and important examples are discussed below). On the other hand, one of the most basic and important objects to investigate in studying a connection on a principal bundle is its holonomy. The starting point for the present work is a study of the holonomy of conformal manifolds, which is defined to be the holonomy of the canonical Cartan connection associated to a conformal structure. The more general aspects from parabolic geometry which are introduced and utilized here, all come into play from pursuing the question of which groups can occur as the holonomy of a conformal manifold.

To get a better picture of how we pose the question which this work is devoted to, it is useful to compare what's known about conformal holonomy with the situation for holonomy of semi-Riemannian manifolds. The holonomy group $Hol(M, g)$ of an oriented semi-Riemannian manifold (M, g) of signature (p, q) is of course a subgroup of $SO(p, q)$. In contrast, if we consider the conformal class of metrics $c = [g]$, the conformal holonomy group $Hol(M, c)$ is a subgroup only of $SO(p + 1, q + 1)$, (cf. Chapter 3.1).

The differences become more interesting if we ask what the geometrical meaning of holonomy-invariant subspaces (under the standard representations) is. First of all, there is a bijective correspondence between one-dimensional $Hol(M, c)$-invariant subspaces of $\mathbb{R}^{p+1,q+1}$ and Einstein metrics in the conformal class c which are defined up to singularities. For decomposable holonomy preserving spaces of larger dimension, there is an analog to the decomposition theorem of De Rham/Wu for semi-Riemannian manifolds, where again the connection to Einstein structures appears and the decomposition holds up to singularities. This and other recent results for reducible conformal holonomy, due to F. Leitner, S. Armstrong and T. Leistner, are detailed in Chapter 3.2.

If on the other hand we look at conformal manifolds with irreducible conformal holonomy, the picture is not so well understood. The first problem is there is no evident analog for conformal holonomy of the Berger list of possible irreducible holonomy groups of semi-Riemannian manifolds. The canonical Cartan connection of conformal geometry is not known to possess any property which, analogous to vanishing torsion for affine connections, produces an algebraic restriction giving a finite list of possible irreducible holonomy groups. Indeed, the possibility of giving a (more or less complete) classification of conformal holonomy groups in Riemannian signature is the result of a very special *algebraic* fact: the only connected irreducible subgroup of $SO(1, n+1)$ (in which the conformal holonomy group of an orientable, conformal Riemannian manifold of dimension n is by definition contained) is the connected component $SO_0(1, n + 1)$. Thus, irreducible conformal holonomy plays no real role in Riemannian signature. Moving to arbitrary signature, the corresponding algebraic result of course no longer holds, and even for $SO(2, n)$ we know of no practically useful (i.e. finite) classification of the irreducible subgroups.

Besides the problem of obtaining a finite list of possible irreducible conformal holonomy groups, the natural question for each such group is what (additional) geometric structures are associated to manifolds with conformal holonomy contained in it, and if there are geometric properties characterizing this. That is, are there analogs to the setting of semi-Riemannian geometry, where irreducible manifolds are divided by holonomy into generic, Kähler, Calabi-Yau, quaternionic Kähler, etc. type and their pseudo-Riemannian variants.

Our Ansatz for studying irreducible conformal holonomies, is to impose the further condition of *transitivity*. That is, we consider only irreducible subgroups H of $SO(p + 1, q + 1)$ which act transitively on the homogeneous model space for conformal geometry $S^{p,q}$ (this space can be thought of as the signature (p, q) analog of the n-sphere, cf. Chapter 3.1 for definition). This new approach allows a systematic study of a broad class of possible irreducible conformal holonomy groups, via parabolic geometries which are naturally connected with them.

In Chapter 3.3 we explain how the generalized Fefferman construction due to Čap in [14] produces out of a parabolic geometry of a certain type (H, Q) a conformal manifold with induced Cartan connection having holonomy naturally contained in H. We also detail an analog of the usual holonomy reduction principle for Cartan connections and, generalizing results from Čap's work on twistor spaces in parabolic geometry [13] to our situation, give conditions for the existence of a local converse to the Fefferman construction for manifolds with conformal holonomy contained in a transitive subgroup H.

There is an extensive theory of transitive transformation groups (cf. A. L. Onishchik's book, [53]), and we make use of results from this theory in Chapter 3.4

to indicate how a list of the possible connected, irreducible transitive conformal holonomy groups in all but one (Lorentzian!) signature may be found. It must be emphasized that the general results in Chapter 3 in no way settle the question of irreducible transitive conformal holonomy. For one thing, the conformal manifold produced from a parabolic geometry of some type (H, Q) by the generalized Fefferman construction is not in general known to have conformal holonomy contained in H. This is because the Cartan connection induced by the Fefferman construction is in general not known to be the canonical Cartan connection of the conformal structure. This is a problem which it appears must be solved by detailed, case-by-case calculations for each of the transitive subgroups of $SO(p + 1, q + 1)$.

The first groups in the list from Chapter 3.4 are familiar from the Berger list. In particular, the first non-generic connected, irreducible transitive conformal holonomy group which can appear as a full holonomy group is known to be $SU(p'+1, q'+1)$. In [18] and [19], Čap and A. R. Gover showed that this conformal holonomy precisely corresponds to (a slight modification of) the classical Fefferman construction, which defines a conformal class of metrics on an S^1 bundle over a CR manifold of signature (p', q'), cf. also [45] for an earlier approach which also considered CR structures with torsion. CR manifolds give the next important example of an underlying geometric structure associated to a parabolic geometry, of type $(PSU(p'+1, q'+1), Q)$ for a certain parabolic subgroup Q. CR structures are discussed in Chapter 4.1, as are their quaternionic analog, the quaternionic contact (QC) structures introduced by O. Biquard in [6]. QC structures are the underlying structures for parabolic geometries of type $(PSp(p'' + 1, q'' + 1), Q')$. The main new result, proved in Chapter 4.3, is a symplectic version of the holonomy correspondence for $SU(p' + 1, q' + 1)$ and CR structures:

Theorem 1 *Given a quaternionic contact structure of signature (p'', q''), the conformal holonomy of the corresponding Fefferman space is contained in $Sp(p''+1, q''+1)$. Conversely, let (M, c) be a conformal manifold of signature $(4p'' + 3, 4q'' + 3)$. If $Hol(M, c)$ is contained in $Sp(p'' + 1, q'' + 1)$, then (M, c) is locally isomorphic to a Fefferman space over a quaternionic contact structure of signature (p'', q'').*

Our proof of this Theorem generalizes the methods of [18] and [19], and we believe this also points to the techniques which could be applied to the other irreducible transitive groups in Chapter 3.4, relying on explicit matrix representations of the groups in question. Finally, Chapter 4.4 gives a simple method of inducing Weyl structures for the conformal Cartan geometry on the Fefferman space. This gives in particular a very direct way, not known to us in the literature, of seeing the relation between the conformal structure of the parabolic Fefferman space over a CR manifold, and the explicit metrics given in the classical Fefferman construction, which is done in Chapter 4.5.

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Chapter 2

Parabolic geometry

Here we develop the language of parabolic Cartan geometries, of which conformal geometry is one of the most important; this will be applied in the subsequent sections to the conformal holonomy problem. The basic references for this material are [21] and [22]. For a survey of the growing field of parabolic geometries, see [14].

2.1 Cartan geometries

Let G be a Lie group and $P \subset G$ a closed subgroup with Lie algebras $\mathfrak{p} \subset \mathfrak{g}$, respectively. Conceptually, a Cartan geometry of type (G, P) is a curved geometry modeled on the homogeneous space G/P. This notion is formalized in the following definition:

Definition 2 *A* Cartan geometry of type (G, P) *is given by the data* $(\mathcal{P}, \pi, M, \omega)$, *where* $\pi : \mathcal{P} \to M$ *is a principal fiber bundle with structure group* P *(or a* P*-PFB) and* ω *is a one-form on* $\mathcal{P}$ *with values in* $\mathfrak{g}$, $\omega \in \Omega^1(\mathcal{P}; \mathfrak{g})$, *called the* Cartan connection *and satisfying, for all* $p \in P, u \in \mathcal{P}$, *and* $X \in \mathfrak{p}$*:*

$$R_p^*(\omega) = \mathrm{Ad}(p^{-1}) \circ \omega; \tag{2.1}$$

$$\omega(\tilde{X}) = X; \tag{2.2}$$

$$\omega(u) : T_u\mathcal{P} \xrightarrow{\cong} \mathfrak{g}. \tag{2.3}$$

In this definition, as in general for any P-PFB, we write $\tilde{X}$ for the fundamental vector field on $\mathcal{P}$ induced by $X \in \mathfrak{p}$. Moreover, by property (2.3), the smooth vector field $\tilde{X}$ on $\mathcal{P}$ given by

$$\tilde{X}(u) = \omega(u)^{-1}(X) \tag{2.4}$$

for all $u \in \mathcal{P}$ is well-defined and non-vanishing for any $X \in \mathfrak{g}$. The Cartan geometry is said to be *complete* if all such vector fields $\tilde{X}$ are complete. In any case, these vector fields give a trivialization of the tangent bundle of $\mathcal{P}$, which we can project to give an isomorphism of the tangent bundle of M with a natural bundle of $\mathcal{P}$:

$$T\mathcal{P} \cong \mathcal{P} \times \mathfrak{g}; \tag{2.5}$$

$$TM \cong \mathcal{P} \times_P (\mathfrak{g}/\mathfrak{p}). \tag{2.6}$$

In particular, we have $dim(M) = dim(G/P)$, a property clearly demanded of a geometry modeled on G/P.

Definition 3 *The* curvature of the Cartan connection ω *is the two-form on* $\mathcal{P}$ *with values in* $\mathfrak{g}$*,* $K^\omega \in \Omega^2(\mathcal{P};\mathfrak{g})$*, defined for all* $u \in \mathcal{P}$*, and all* $\xi, \eta \in T_u\mathcal{P}$*, by:*

$$K^\omega(\xi,\eta) = d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)]_{\mathfrak{g}}. \tag{2.7}$$

From the properties (2.1) and (2.2) of the Cartan connection, the curvature of a Cartan connection of type (G,P) is P-equivariant and horizontal. Thus it induces a two-form on M which we'll denote with the same symbol, $K^\omega \in \Omega^2(M;\mathfrak{g})$.

Example 4 Let G be a Lie group, P a closed subgroup, and $\pi : G \to G/P$ the projection onto the quotient. For any $g \in G$ and $\xi \in T_gG$, let $\omega^G \in \Omega^1(G,\mathfrak{g})$ be the Maurer-Cartan form defined by

$$\omega^G(\xi) = (L_{g^{-1}})_*(\xi) \in T_eG = \mathfrak{g}.$$

Then $(G,\pi,G/P,\omega^G)$ defines a Cartan geometry of type (G,P). By the structure equation for the Maurer-Cartan form, its curvature K^{ω^G} vanishes identically. □

So it really is legitimate to think of Cartan geometries as curved generalizations of homogeneous spaces. The following, which could be called the fundamental theorem of Cartan geometry, says that the curvature of the Cartan connection really measures how much a Cartan geometry deviates from the homogeneous model space:

Theorem 5 *The curvature of a Cartan geometry of type* (G,P) *vanishes identically if and only it is locally isomorphic to the homogeneous model geometry in Example 4. If the Cartan geometry is complete, with connected structure group and simply connected base manifold, the isomorphism is global.*

For a proof, see Chapter 5.5 of [56].

For any Cartan geometry, there is a special class of natural vector bundles, the so-called Tractor bundles. Given a representation $\rho : G \to GL(W)$ of the group G, the restriction of the representation to the subgroup P defines the *associated Tractor bundle*:

$$\mathcal{W} = \mathcal{P} \times_{(P,\rho)} W. \tag{2.8}$$

An important property of associated Tractor bundles is that they come with naturally induced linear connections. This can be seen via the *principal Tractor bundle* of a Cartan geometry $(\mathcal{P},\pi,M,\omega)$ of type (G,P), which is a G-PFB containing $\mathcal{P}$ together with a principal bundle connection. Define *the extension of* $\mathcal{P}$ to a G-PFB by

$$\mathcal{G} := \mathcal{P} \times_P G.$$

Then

$$\begin{aligned} \iota &: \mathcal{P} \hookrightarrow \mathcal{G} \\ \iota &: u \mapsto [u,e] \end{aligned}$$

gives an inclusion, and there exists a unique principal bundle connection $\bar{\omega} \in \Omega^1(\mathcal{G},\mathfrak{g})$ such that $\iota^*\bar{\omega} = \omega$. We call the set $(\mathcal{G},\bar{\pi},M,\bar{\omega})$ the principal Tractor bundle.

Then for any associated Tractor bundle defined by a representation $\rho : G \to GL(W)$, we have

$$\begin{aligned}\mathcal{W} &= \mathcal{P} \times_{(P,\rho)} W \\ &\cong \mathcal{G} \times_{(G,\rho)} W,\end{aligned}$$

and hence $\mathcal{W}$ inherits a linear connection $\nabla^{\mathcal{W}}$ from $\bar{\omega}$ in the usual manner.

2.2 Parabolic subgroups and graded Lie algebras

Cartan geometries include a very broad range of geometric structures, but in some ways this is a drawback. For an arbitrary pair (G, P), it's a very hard problem to determine a natural Cartan geometry of type (G, P), and the geometric meaning of the Cartan connection may be just as difficult to understand. For the sub-class of parabolic Cartan geometries, the situation is much better. The important properties of parabolic geometries which will be developed in the next section depend crucially on the algebraic facts about parabolic subgroups, which we review here. References for most of this material are Chapter 2 of [3] and Chapter 2 of [49].

Definition 6 *A* parabolic pair *is a pair of groups (G, P), where G is a (real or complex) semi-simple Lie group, and $P \subset G$ is a* parabolic subgroup. *That is, the Lie algebra $\mathfrak{p}$ of P is* parabolic. *I.e. in the complex case it contains a maximal solvable (or Borel) subalgebra of the Lie algebra $\mathfrak{g}$ of G; in the real case, the complexification has this property. Such a pair of Lie algebras $(\mathfrak{g}, \mathfrak{p})$ is called an* infinitesimal parabolic pair.

Parabolic pairs are closely related to graded Lie algebras:

Definition 7 *A* $|k|$-grading of a semi-simple Lie algebra $\mathfrak{g}$, *for $k \in \mathbb{N}$, is a vector space decomposition*

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, \tag{2.9}$$

which is compatible with the Lie bracket:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \tag{2.10}$$

where by definition, $\mathfrak{g}_l := \{0\}$ for $|l| > k$. If furthermore the nilpotent subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$, and no simple factor of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, then the graded Lie algebra is called effective.

For an effective $|k|$-graded semi-simple Lie algebra $\mathfrak{g}$ and for $-k \leq i \leq k$, letting $\mathfrak{g}^i = \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k$ defines a filtration

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \ldots \supset \mathfrak{g}^0 \supset \ldots \supset \mathfrak{g}^k \supset 0 \tag{2.11}$$

which is compatible with the Lie bracket. Now, given a Lie group G with $\mathfrak{g}$ as its Lie algebra, let G_0 be the subgroup of all elements which preserve the grading of $\mathfrak{g}$ and let P be the subgroup of all elements preserving the induced filtration. Then the Lie algebra of G_0 is $\mathfrak{g}_0$, the Lie algebra of P is $\mathfrak{p} = \mathfrak{g}^0$, and $\mathfrak{p}$ is a parabolic subalgebra.

Furthermore, the subalgebra $\mathfrak{p}_+ := \mathfrak{g}^1$ determines a subgroup $P_+ \subset P$ which is diffeomorphic to $\mathfrak{p}_+$ under the exponential map, and $G_0 \cong P/P_+$ so $P = G_0 \rtimes P_+$. The structure of P is further described by the following Proposition (cf. Proposition 2.10 of [21]):

Proposition 8 *Let G be a semisimple Lie group whose Lie algebra $\mathfrak{g}$ has a $|k|$-grading, and let P be the corresponding parabolic subgroup as above. Then for any element $g \in P$, there exist unique elements $g_0 \in G_0$ and $X_i \in \mathfrak{g}_i$ for $i = 1, \ldots, k$, such that $g = g_0 \exp(X_1) \ldots \exp(X_k)$.*

The following Proposition (cf. Proposition 2.2 of [21]) summarizes a number of other important basic properties we'll need:

Proposition 9 *Let $\mathfrak{g}$ be an effective semisimple $|k|$-graded Lie algebra. Then the following assertions hold:*
1. There is a unique element $\varepsilon_0 \in \mathfrak{g}_0$, called the grading element, such that $[\varepsilon_0, X] = jX$ for all $X \in \mathfrak{g}_j$.
2. Let $B_\mathfrak{g}$ be the Killing form. Then $B_\mathfrak{g}(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i+j = 0$, and $B_\mathfrak{g}$ induces an isomorphism $\mathfrak{g}_i \cong (\mathfrak{g}_{-i})^*$ of $\mathfrak{g}_0$-modules for all $i = 1, \ldots, k$.

To see how on the other hand a parabolic pair defines a graded Lie algebra, we note first that a parabolic sub-algebra $\mathfrak{p}$ of a semi-simple Lie algebra $\mathfrak{g}$ automatically induces a filtration of $\mathfrak{g}$. For a subspace $\mathfrak{u} \subset \mathfrak{g}$, let $\mathfrak{u}^\perp$ be the orthogonal subspace with respect to the Killing form $B_\mathfrak{g}$. The following fact is standard:

Lemma 10 *For a parabolic subalgebra $\mathfrak{p}$ of a semi-simple Lie algebra $\mathfrak{g}$, $\mathfrak{p}_+ := \mathfrak{p}^\perp$ is the maximal nilpotent ideal of $\mathfrak{p}$. The quotient $\mathfrak{p}_0 := \mathfrak{p}/\mathfrak{p}_+$ is reductive and $\mathfrak{p} \cong \mathfrak{p}_0 \oplus \mathfrak{p}_+$.*

Definition 11 *For an infinitesimal parabolic pair $(\mathfrak{g}, \mathfrak{p})$ as above, let*

$$\mathfrak{p}_+, (\mathfrak{p}_+)^2 = [\mathfrak{p}_+, \mathfrak{p}_+], \ldots, (\mathfrak{p}_+)^k = [\mathfrak{p}_+, (\mathfrak{p}_+)^{k-1}] \neq 0, (\mathfrak{p}_+)^{k+1} = 0 \tag{2.12}$$

be the descending central series of $\mathfrak{p}_+$. Let $\mathfrak{g}^0 := \mathfrak{p}$ and $\mathfrak{g}^j := (\mathfrak{p}_+)^j$ for $1 \leq j$. For j positive, define $\mathfrak{g}^{-j} := (\mathfrak{g}_{j+1})^\perp$. Then $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ and

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \ldots \supset \mathfrak{g}^0 = \mathfrak{p} \supset \mathfrak{p}_+ = \mathfrak{g}^1 \supset \ldots \supset \mathfrak{g}^k \supset 0 \tag{2.13}$$

is the Lie algebra filtration of $\mathfrak{g}$ for the pair $(\mathfrak{g}, \mathfrak{p})$.

The *associated graded algebra* to the filtered Lie algebra $\mathfrak{g}$ is given by

$$\mathrm{gr}(\mathfrak{g}) := (\mathfrak{g}^{-k}/\mathfrak{g}^{-k+1}) \oplus \ldots \oplus (\mathfrak{g}^{-1}/\mathfrak{g}^0) \oplus (\mathfrak{g}^0/\mathfrak{g}^1) \oplus (\mathfrak{g}^1/\mathfrak{g}^2) \oplus \ldots \oplus (\mathfrak{g}^k) \tag{2.14}$$

$$=: \mathrm{gr}_{-k}(\mathfrak{g}) \oplus \ldots \oplus \mathrm{gr}_{-1}(\mathfrak{g}) \oplus \mathrm{gr}_0(\mathfrak{g}) \oplus \mathrm{gr}_1(\mathfrak{g}) \oplus \ldots \oplus \mathrm{gr}_k(\mathfrak{g}). \tag{2.15}$$

While the filtration of $\mathfrak{g}$ is canonical given a parabolic subalgebra, a choice is required to identify $\mathfrak{g}$ with the graded Lie algebra, cf. Lemma 2.2 of [12]:

Lemma 12 *There are (non-canonical) splittings of the exact sequences*

$$0 \to \mathfrak{g}^{j+1} \to \mathfrak{g}^j \to \mathrm{gr}_j(\mathfrak{g}) \to 0 \tag{2.16}$$

which induce a Lie algebra isomorphism between $\mathfrak{g}$ and $\mathrm{gr}\mathfrak{g}$.

Such a choice of splittings is an *algebraic Weyl structure.* An algebraic Weyl structure may be fixed, in the complex case, via a choice of a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$, a root system $\Delta(\mathfrak{g}, \mathfrak{c})$ for it, and a set of simple, positive roots $\Delta^0 \subset \Delta^+$. Then a standard parabolic sub-algebra may be defined for any subset of Δ^0:

Definition 13 *Given a subset $\Sigma \subseteq \Delta^0$ of simple, positive roots, the* standard parabolic subalgebra $\mathfrak{p}_\Sigma$ of Σ *is given by*

$$\mathfrak{p}_\Sigma = \mathfrak{c} \oplus < \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha > + < \bigoplus_{\alpha \in \Delta^0 \setminus \Sigma} \mathfrak{g}_{-\alpha} > \tag{2.17}$$

In particular, the set $\Sigma = \Delta^0$ of all simple roots corresponds to the *standard Borel subalgebra* for that choice of Cartan subalgebra and positive roots. It is a standard result in the structure theory of semi-simple Lie algebras that all Borel (i.e. maximal solvable) subalgebras are conjugate, and hence conjugate to a standard one, implying that every parabolic subalgebra is also conjugate to a standard parabolic determined by a subset Σ as in Definition 13.

Standard parabolics have two important and nice properties. First, as is clear from Definition 13, standard (complex) parabolics may be determined by marked Dynkin diagrams, namely by the Dynkin diagram of the semi-simple Lie algebra $\mathfrak{g}$, with crosses over those nodes corresponding to the simple positive roots contained in Σ. Furthermore, given a standard parabolic subalgebra determined by a set of roots Σ, $\mathfrak{g}$ receives a grading given by Σ-height:

Definition 14 *For Σ as above,* the Σ-height of a root α, $\mathrm{ht}_\Sigma(\alpha)$ *is the sum of coefficients of roots in Σ in the representation of α in terms of simple roots.*

For a standard parabolic pair $(\mathfrak{g}, \mathfrak{p}_\Sigma)$, the grading components of $\mathfrak{g}$ are given by:

$$\mathfrak{g}_0 = \mathfrak{c} \oplus < \bigoplus_{\mathrm{ht}_\Sigma(\alpha)=0} \mathfrak{g}_\alpha >; \tag{2.18}$$

$$\mathfrak{g}_i =< \bigoplus_{\mathrm{ht}_\Sigma(\alpha)=i} \mathfrak{g}_\alpha >, \ \forall\, i \in \mathbb{Z}, i \neq 0. \tag{2.19}$$

Then for some integer $k \geq 1$,

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k \tag{2.20}$$

determines a $|k|$-grading of $\mathfrak{g}$ and a Lie algebra isomorphism of $\mathfrak{g}$ with $\mathrm{gr}\mathfrak{g}$*, the associated graded algebra determined by the canonical filtration of $\mathfrak{g}$ given by the parabolic.*

We note also that these notions can be carried over, with some adjustments, to parabolic subalgebras of real semi-simple Lie algebras, which may be described in terms of Satake diagrams, cf. [58]. For our purposes, i.e. for the parabolic geometries which interest us in Chapters 3 and 4, the parabolic subalgebra may always be taken as standard since we will work with explicit matrix representations of the Lie algebras, and the gradings will be given explicitly. Furthermore, the Lie algebras we'll deal with are all simple and their gradings effective. For these reasons, in the sequel and in the following sections on parabolic geometries, we will always take as given an effective semi-simple $|k|$-graded (real or complex) Lie algebra. A parabolic pair (G, P) with Lie algebras satisfying these conditions will be called *standard, effective.*

We conclude this section with facts we'll need from Lie algebra cohomology theory, applied to graded Lie algebras. As discussed above, we have a graded algebra $\mathfrak{g}$. In particular, we have the decompositions:

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p} \tag{2.21}$$

$$= \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \tag{2.22}$$

where $\mathfrak{g}_+ := \mathfrak{g}^1 = \mathfrak{p}_+$ and the Killing form $B_\mathfrak{g}$ induces an isomorphism $\mathfrak{g}_+ \cong (\mathfrak{g}_-)^*$. Note also the isomorphism $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_-$, with which $\mathfrak{g}_-$ is endowed with a P-module structure. We are interested in the cohomology of the module $\mathfrak{g}$, under the adjoint representation restricted to the subalgebras $\mathfrak{g}_-$ or $\mathfrak{g}_+$. Then the chain groups and differentials are given by:

Definition 15 *The* chain groups $C^n(\mathfrak{g}_\mp, \mathfrak{g})$ *of the graded Lie algebra* $\mathfrak{g}$ *are*

$$C^n(\mathfrak{g}_\mp, \mathfrak{g}) := \mathrm{Hom}(\Lambda^n \mathfrak{g}_\mp, \mathfrak{g}). \tag{2.23}$$

The differential $\partial : C^n(\mathfrak{g}_\mp, \mathfrak{g}) \to C^{n+1}(\mathfrak{g}_\mp, \mathfrak{g})$ *is given by*

$$\begin{aligned}(\partial\phi)(X_0, \dots, X_n) := &\sum_{i=0}^{n} (-1)^i [X_i, \phi(X_0, \dots, \hat{X}_i, \dots, X_n)] \\ &+ \sum_{0 \le i < j \le n} (-1)^{i+j} \phi([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_n),\end{aligned}$$

for $\phi \in C^n(\mathfrak{g}_\mp, \mathfrak{g})$ *and* $X_0, \dots, X_n \in \mathfrak{g}_\mp$.

It is a basic result from cohomology of Lie algebra modules that the pairs $(C^*(\mathfrak{g}_\mp, \mathfrak{g}), \partial)$ form chain complices, i.e. $\partial \circ \partial = 0$. Moreover, the chain groups inherit natural G_0-module structures (in fact, they have P-module structures) and the differentials are G_0-module homomorphisms. The cohomology groups also have the structure of G_0-modules; they are defined by

$$H^n(\mathfrak{g}_\mp, \mathfrak{g}) = \frac{\ker(\partial : C^n(\mathfrak{g}_\mp, \mathfrak{g}) \to C^{n+1}(\mathfrak{g}_\mp, \mathfrak{g}))}{\mathrm{im}(\partial : C^{n-1}(\mathfrak{g}_\mp, \mathfrak{g}) \to C^n(\mathfrak{g}_\mp, \mathfrak{g}))}. \tag{2.24}$$

Furthermore, we denote by $C^n_l(\mathfrak{g}_-, \mathfrak{g})$ the space of chains of homogeneity l:

$$C^n_l(\mathfrak{g}_-, \mathfrak{g}) := \{\phi \in C^n(\mathfrak{g}_-, \mathfrak{g}) | \phi(\mathfrak{g}_{i_1}, \dots, \mathfrak{g}_{i_n}) \subseteq \mathfrak{g}_{i_1+\dots+i_n+l}\}. \tag{2.25}$$

From the definition, it follows that the differential ∂ preserves homogeneity, and thus the cohomology groups split:

$$H^n(\mathfrak{g}_-, \mathfrak{g}) = \oplus_l H^n_l(\mathfrak{g}_-, \mathfrak{g}). \tag{2.26}$$

Finally, we note that a *codifferential* ∂^* can be defined using the isomorphisms

$$\begin{aligned}C^n(\mathfrak{g}_-, \mathfrak{g}) &\cong \Lambda^n(\mathfrak{g}_-)^* \otimes \mathfrak{g} \\ &\cong (\Lambda(\mathfrak{g}_+)^* \otimes \mathfrak{g}^*)^* \cong (C^n(\mathfrak{g}_+, \mathfrak{g}^*))^*.\end{aligned}$$

$\partial^* : C^n(\mathfrak{g}_-, \mathfrak{g}) \to C^{n-1}(\mathfrak{g}_-, \mathfrak{g})$ is defined to be the negative of the dual map to $\partial : C^{n-1}(\mathfrak{g}_+, \mathfrak{g}^*) \to C^n(\mathfrak{g}_+, \mathfrak{g}^*)$. The codifferential and cohomology have the following important properties (cf. 2.5, Proposition 2.6 and Proposition 2.13 of [21]):

Proposition 16 *There exists a metric on the chain groups with respect to which* ∂^* *is adjoint to* ∂*. In particular, we have the Hodge decomposition*

$$C^n(\mathfrak{g}_-, \mathfrak{g}) = \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*) \oplus (\ker(\partial) \cap \ker(\partial^*)), \tag{2.27}$$

and for the Kostant Laplace operator $\Box = \partial \circ \partial^* + \partial^* \circ \partial$*, this decomposition induces an isomorphism*

$$H^n(\mathfrak{g}_-, \mathfrak{g}) \cong \ker(\Box). \tag{2.28}$$

The codifferential $\partial^* : C^2(\mathfrak{g}_-, \mathfrak{g}) \to C^1(\mathfrak{g}_-, \mathfrak{g})$ *can be expressed as follows. Let* $\{X_\alpha\}$ *be a basis for* $\mathfrak{g}_-$ *and let* $\{Z_\alpha\}$ *be a dual basis of* $\mathfrak{g}_+$ *with respect to the Killing form. Then for* $\phi \in C^2(\mathfrak{g}_-, \mathfrak{g})$ *and* $X \in \mathfrak{g}_-$*, we have:*

$$(\partial^*\phi)(X) = \sum_\alpha [\phi(X, X_\alpha), Z_\alpha] - \frac{1}{2} \sum_\alpha \phi([X, Z_\alpha]_-, X_\alpha). \tag{2.29}$$

The codifferential $\partial^* : C^2(\mathfrak{g}_-, \mathfrak{g}) \to C^1(\mathfrak{g}_-, \mathfrak{g})$ *is a homomorphism of* P*-modules.*

The P-equivariance of the codifferential is essential for many of the natural constructions with parabolic geometries. In particular, the existence of *harmonic prolongations* (cf. Section 3 of [21]) depends on this property, which in turn are necessary to establish the existence of canonical Cartan geometries for parabolic structure, as discussed in the sequel.

Especially for the calculations in Chapter 4.2, it is often useful to write the codifferential $\partial^* : C^2(\mathfrak{g}_-, \mathfrak{g}) \to C^1(\mathfrak{g}_-, \mathfrak{g})$ as the sum of two operators $\partial_1^* - \partial_2^*$ from the formula (2.29): For $X \in \mathfrak{g}_-$ and $\phi \in C^2(\mathfrak{g}_-, \mathfrak{g})$, and $\{X_\alpha\}, \{Z_\alpha\}$ as above, we define:

$$(\partial^*\phi)_1(X) := \sum_\alpha [\phi(X, X_\alpha), Z_\alpha]; \tag{2.30}$$

$$(\partial^*\phi)_2(X) := \frac{1}{2}\sum_\alpha \phi([X, Z_\alpha]_-, X_\alpha). \tag{2.31}$$

2.3 Basic properties of parabolic geometries

Definition 17 *A* parabolic geometry *is a Cartan geometry* $(\mathcal{P}, \pi, M, \omega)$ *of type* (G, P)*, for a standard, effective parabolic pair* (G, P)*.*

Parabolic geometries have the nice feature, not evident for general Cartan geometries, that they can be understood in terms of "normal" geometric structures on the base manifold M. Let us first take note of the induced filtration on the tangent bundle TM. This is induced by a canonical filtration on the so called adjoint bundle, an important object in its own right:

Definition 18 *The* adjoint bundle *of a parabolic geometry* $(\mathcal{P}, \pi, M, \omega)$ *is the associated Tractor bundle to the* P*-PFB determined by the adjoint representation* $(Ad, \mathfrak{g})$ *of* G*, restricted to* P*:*

$$\mathcal{A}(M) = \mathcal{P} \times_{(P,Ad)} \mathfrak{g}. \tag{2.32}$$

$\mathcal{A}(M)$ *has a natural bracket* $\{,\}_\mathcal{A}$ *induced pointwise by the bracket of* $\mathfrak{g}$*. Moreover,* P*-invariance of the filtration (2.11) allows us to translate this filtration to* $\mathcal{A}(M)$*, giving it the structure of a filtered Lie algebra bundle. Letting* $\mathcal{A}^i(M) = \mathcal{P} \times_{(P,Ad)} (\mathfrak{g}^i)$*, for* $-k \leq i \leq k$*, the filtration is given by:*

$$\mathcal{A}(M) = \mathcal{A}^{-k}(M) \supset \mathcal{A}^{-k+1}(M) \supset \ldots \supset \mathcal{A}^k(M) \supset \{0\}. \tag{2.33}$$

From this filtration, we can define the *associated graded adjoint bundle* $\mathrm{gr}\mathcal{A}(M)$:

$$\mathrm{gr}\mathcal{A}(M) := (\mathcal{A}^{-k}(M)/\mathcal{A}^{-k+1}(M)) \oplus \ldots \oplus (\mathcal{A}^k(M)/\{0\}) \tag{2.34}$$

$$=: \mathrm{gr}_{-k}\mathcal{A}(M) \oplus \ldots \oplus \mathrm{gr}_k\mathcal{A}(M). \tag{2.35}$$

Now, from the identification of the tangent bundle of M in (2.6) for general Cartan geometries, and the P-module isomorphism $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_-$ for our graded algebra, we obtain from the quotient map

$$\begin{aligned}\mathcal{A}(M) \to &(\mathcal{A}(M)/\mathcal{A}^0(M))\\ &\cong \mathcal{A}_-(M) = \mathcal{P} \times_P \mathfrak{g}_-\end{aligned}$$

a natural projection

$$\Pi : \mathcal{A}(M) \to TM. \tag{2.36}$$

This induces a filtration of TM, letting $T^iM = \Pi(\mathcal{A}^i(M))$:

$$TM = T^{-k}M \supset T^{-k+1}M \supset \ldots T^{-1}M \supset T^0M = \{0\}. \tag{2.37}$$

The tangent bundle $T\mathcal{P}$ also has a natural filtration for a parabolic geometry, and we note that the above described filtration on TM can equivalently be defined via the projection $T\mathcal{P} \to TM$.

From the filtration (2.37), we of course can also define the *associated graded tangent bundle* $\mathrm{gr}(TM)$:

$$\mathrm{gr}(TM) := (T^{-k}M/T^{-k+1}M) \oplus \ldots \oplus (T^{-1}M/\{0\}) \tag{2.38}$$
$$=: \mathrm{gr}_{-k}(TM) \oplus \ldots \oplus \mathrm{gr}_{-1}(TM). \tag{2.39}$$

Besides these filtrations given by a parabolic geometry $(\mathcal{P}, \pi, M, \omega)$, we also have a G_0-PFB $\pi_0 : \mathcal{G}_0 \to M$, defined by $\mathcal{G}_0 := \mathcal{P}/P_+$. The grading of $\mathfrak{g}$ gives rise to a decomposition of ω, which allows us to define an algebraic bracket on $\mathrm{gr}(TM)$ and to identify $\mathcal{G}_0$ as a reduction of the adapted frame bundle of $\mathrm{gr}(TM)$ to the structure group G_0:

Definition 19 *Consider the decomposition of the Cartan connection*

$$\omega = \omega_{-k} + \ldots + \omega_k \tag{2.40}$$

given by the $|k|$-grading on $\mathfrak{g}$. The soldering pre-form *induced by the parabolic geometry is a k-tuple θ of partially defined one-forms on $\mathcal{G}_0$:*

$$\theta = (\theta_{-k}, \ldots, \theta_{-1}), \tag{2.41}$$

where each $\theta_i \in \Gamma((T^i\mathcal{G}_o)^* \otimes \mathfrak{g}_i)$ is defined as follows for $-k \leq i \leq -1$: For $\xi_0 \in T^i_{u_0}\mathcal{G}_0$, choose $\xi \in T^i_u\mathcal{P}$ with $(\pi_+)_*\xi = \xi_0$, where $\pi_+ : \mathcal{P} \to \mathcal{G}_0$ is the obvious projection. Then

$$\theta_i(\xi_0) := \omega_i(\xi). \tag{2.42}$$

It follows from the properties of the Cartan connection that the soldering pre-form θ is well-defined. Moreover, it is G_0-equivariant and horizontal (cf. Sections 3.2 and 4.1 of [21]), and so induces a k-tuple of partially defined one-forms on M. Each component θ_i of the soldering pre-form induces a linear isomorphism, for all $x \in M$:

$$\theta_i : T^i_xM/T^{i+1}_xM \xrightarrow{\cong} \mathfrak{g}_i. \tag{2.43}$$

This gives a reduction to G_0 of the structure group of the associated graded tangent bundle $\mathrm{gr}(TM)$. An algebraic bracket $\{,\}_{\mathfrak{g}_0}$ is defined on $\mathrm{gr}(TM)$ via this G_0-structure. Explicitly, for $X \in (T^i_xM/T^{i+1}_xM)$ and $Y \in (T^j_xM/T^{j+1}M)$, define

$$\{X, Y\}_{\mathfrak{g}_0} := \theta^{-1}_{i+j}([\theta_i(X), \theta_j(Y)]_{\mathfrak{g}}) \in (T^{i+j}M/T^{i+j+1}M) \tag{2.44}$$

The problem is the bracket defined by (2.44) does not have an evident, natural meaning in terms of a geometry on M. On the other hand, there is a tensorial map, called the *generalized Levi-form*, defined as follows: Let $x \in M$, $X_x \in T^i_xM$ and $Y_x \in T^j_xM$ for $-k \leq i, j \leq -1$. Then take X and Y to be local vector fields extending X_x and Y_x, respectively. Then

$$L : T^i_xM \otimes T^j_xM \to T_xM/T^{i+j+1}_xM, \tag{2.45}$$
$$L : (X_x, Y_x) \mapsto [X, Y](x) + T^{i+j+1}_xM \tag{2.46}$$

is a well-defined map, determining the generalized Levi-form.

In order to relate this naturally defined bracket on the filtered bundle TM to the algebraic bracket, we need to make use of natural decompositions of the Cartan curvature of a parabolic geometry. First, note that the adjoint bundle and its natural projection (2.36) to the tangent bundle, allow us to identify the Cartan curvature form $K^\omega \in \Omega^2(M;\mathfrak{g})$ with a P-equivariant function in the P-module $C^2(\mathfrak{g}_-,\mathfrak{g})$. We have:

$$\Omega^2(M;\mathfrak{g}) = \Gamma(\Lambda^2(T^*M)\otimes(\mathcal{P}\times_P\mathfrak{g})) \cong \Gamma(\mathcal{P}\times_P(\Lambda^2(\mathfrak{g}/\mathfrak{p})^*\otimes\mathfrak{g})).$$

Thus the curvature form K^ω corresponds to a P-equivariant smooth function on $\mathcal{P}$ with values in the P-module $\Lambda(\mathfrak{g}/\mathfrak{p})^*\otimes\mathfrak{g} = C^2(\mathfrak{g}/\mathfrak{p},\mathfrak{g})$. This function, which we'll denote by κ^ω, can also be explicitly determined, for all $X,Y\in\mathfrak{g}/\mathfrak{p}$, by:

$$\kappa^\omega(u)(X,Y) = K^\omega(\omega(u)^{-1}(X),\omega(u)^{-1}(Y)). \tag{2.47}$$

Definition 20 *For a parabolic geometry, consider the decomposition of the curvature function κ on $\mathcal{P}$ with values in $\Lambda^2(\mathfrak{g}_-)^*\otimes\mathfrak{g}$,*

$$\kappa = \kappa_- + \kappa_0 + \kappa_+, \tag{2.48}$$

given by the decomposition $\mathfrak{g} = \mathfrak{g}_-\oplus\mathfrak{g}_0\oplus\mathfrak{g}_+$. The Cartan connection is said to be torsion-free *if κ_- vanishes identically.*

Now consider the decomposition of κ by homogeneity,

$$\kappa = \sum_{l=-k+2}^{3k}\kappa^{(l)}, \tag{2.49}$$

where $\kappa^{(l)}$ has homogeneity l, i.e. $\kappa^{(l)}(u)(\mathfrak{g}_i,\mathfrak{g}_j)\subset\mathfrak{g}_{i+j+l}$. The Cartan connection is said to be regular *if its curvature has strictly positive homogeneity, i.e. if $\kappa^{(l)}=0$ for all $l\leq 0$.*

The Cartan connection is normal *if*

$$\partial^*\circ\kappa = 0, \tag{2.50}$$

where $\partial^*: C^2(\mathfrak{g}_-,\mathfrak{g})\to C^1(\mathfrak{g}_-,\mathfrak{g})$ is the codifferential given by the formula (2.29).

Note, in particular, that a torsion-free connection is automatically regular. The following Lemma (cf. Lemma 2.7 of [22]) clarifies the relationship between the brackets introduced above and shows the role the Cartan connection and its curvature play:

Lemma 21 *Let $(\mathcal{P},\pi,M,\omega)$ be a parabolic geometry with curvature κ. The generalized Levi-form L defined by (2.46) is compatible with the filtration of the tangent bundle (2.37) if and only if $\kappa^{(l)}$ vanishes for all $l<0$. Moreover, the bracket induced on* $\mathrm{gr}(TM)$ *by the Levi-form coincides with algebraic bracket $\{,\}_{\mathfrak{g}_0}$ if and only if the Cartan connection is regular.*

In summary, a parabolic geometry $(\mathcal{P},\pi,M,\omega)$ of type (G,P) with regular Cartan connection induces what is called a regular infinitesimal flag structure of type $(\mathfrak{g},\mathfrak{p})$ on the base manifold M (cf. 2.7 of [22]):

Definition 22 *Given a standard, effective parabolic pair* (G, P) *and a manifold* M *of dimension* $n = dim(\mathfrak{g}/\mathfrak{p})$, *a* regular infinitesimal flag structure of type $(\mathfrak{g}, \mathfrak{p})$ *is a filtration of the tangent bundle* TM *together with a reduction of the structure group of the associated graded tangent bundle* $\mathrm{gr}(TM)$ *to* $G_0 \cong P/P_+$, *giving a pointwise Lie algebra isomorphism of* $(\mathrm{gr}(TM), L)$ *with* $\mathfrak{g}_- \cong \mathfrak{g}/\mathfrak{p}$.

The fundamental theorem of parabolic geometry says that for essentially all standard, effective parabolic pairs (G, P), there is a converse to the construction given above, associating to every regular infinitesimal flag structure a parabolic geometry which is canonical up to isomorphism:

Theorem 23 *Let* (G, P) *be a standard, effective parabolic pair with* $Z(G)$ *trivial, and such that* $H^1_l(\mathfrak{g}_-, \mathfrak{g})$ *is trivial for all* $l > 0$. *Then there is a bijective correspondence between the isomorphism classes of regular, normal parabolic geometries of type* (G, P) *and regular infinitesimal flag structures of type* $(\mathfrak{g}, \mathfrak{p})$ *on* M.

For a proof of this Theorem, see Section 3 of [21], where the authors also indicate how to handle the exceptional parabolic geometries corresponding to those pairs with non-vanishing positive first cohomology group. The class of regular, normal parabolic geometries satisfying the premises of the Theorem as stated is however sufficiently large for our purposes. Among others, it includes conformal structures (see Chapter 3.1) as well as CR-structures and quaternionic contact (QC) structures (see Chapter 4.1).

Finally, we mention that the restriction that G have trivial center in the Theorem may also be weakened to groups with finite center. Either one may work locally, in which case any such G may be considered by taking finite coverings, or else globally by introducing topological restrictions on M. We will see how this is dealt with for the particular geometries dealt with in the sequel.

2.4 Weyl structures

The original notion of a Weyl structure for general parabolic geometries is due to Čap and J. Slovák in [22]; a different but (in our setting) equivalent approach was later developed in [12]. We will make use of both, motivating the structures at first in the sense of [12].

An algebraic Weyl structure for a parabolic pair (G, P) – defined as a choice of splitting of the sequences (2.16) – may equivalently be viewed as a choice of lift of the grading element. Consider, in particular the exact sequence from the "middle" of the filtration:

$$0 \to \mathfrak{p}_+ \to \mathfrak{p} \overset{\pi_+}{\to} (\mathfrak{p}/\mathfrak{p}_+) \to 0 \tag{2.51}$$

$$= 0 \to \mathfrak{g}^1 \to \mathfrak{g}^0 \overset{\pi_+}{\to} \mathfrak{g}_0 \to 0. \tag{2.52}$$

Then since $\mathrm{gr}(\mathfrak{g})$ is graded, there exists a unique grading element $\varepsilon_0 \in \mathfrak{g}_0$. Let

$$\mathfrak{w} := \{\varepsilon \in \mathfrak{g}^0 | \pi_+(\varepsilon) = \varepsilon_0\} \tag{2.53}$$

be the set of all lifts of the grading element with respect to (2.52).

Proposition 24 *(cf. Lemma 2.5 of [12]) Each element of* $\mathfrak{w}$ *is precisely an algebraic Weyl structure. Moreover, the group* $P_+ = \exp \mathfrak{p}_+$ *acts freely and transitively on* $\mathfrak{w}$.

A (geometric) Weyl structure for a parabolic geometry $(\mathcal{P}, \pi, M, \omega)$ of type (G, P) is, conceptually, the "bundle-ization" of a choice of algebraic Weyl structure. Since the grading element $\varepsilon_0 \in \mathfrak{p}/\mathfrak{p}_+ = \mathfrak{g}_0$ is canonical, it induces a canonical grading section E_0 of the grade zero component

$$\mathrm{gr}_0\mathcal{A}(M) = \mathcal{P} \times_P (\mathfrak{p}/\mathfrak{p}_+)$$

of the associated graded adjoint bundle $\mathrm{gr}\mathcal{A}(M)$.

Definition 25 *(cf. Definition 3.2 of [12]) A (geometric)* Weyl structure E *on* M *is a smooth lift of* E_0 *to a section of* $\mathcal{A}^0(M)$*, with respect to the exact sequence:*

$$0 \to \mathcal{A}^1(M) \to \mathcal{A}^0(M) \to \mathrm{gr}_0\mathcal{A}(M) \to 0. \tag{2.54}$$

Note that this sequence really is the bundle-ization of the algebraic sequence (2.52):

$$(2.54) = \mathcal{P} \times_{(P,\mathrm{Ad})} (0 \to \mathfrak{g}^1 \to \mathfrak{g}^0 \to \mathfrak{g}_0 \to 0).$$

From Proposition 24, it is evident that a choice of algebraic Weyl structure may also be bundle-ized by choosing a P-invariant smooth function

$$\mathcal{E} : \mathcal{P} \to \mathfrak{w}. \tag{2.55}$$

A function $\mathcal{E}$ as in (2.55) induces a P-equivariant isomorphism

$$\mathcal{E}_\bullet : \mathcal{P} \times (\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+) \to \mathcal{P} \times \mathfrak{g}, \tag{2.56}$$

which – recalling the isomorphisms $T^*M \cong \mathcal{A}^1(M)$ and $TM \cong \mathcal{A}(M)/\mathcal{A}^0(M)$ – in turn induces a bundle isomorphism:

$$E_\bullet : TM \oplus \mathrm{Gr}_0\mathcal{A}(M) \oplus T^*M \to \mathcal{A}(M). \tag{2.57}$$

On the other hand, if an algebraic Weyl structure ε for (G, P) is fixed, then a function $\mathcal{E}$ as in (2.55) may be written as $\mathcal{E} = (\mathrm{Ad}q)\varepsilon$ for a uniquely determined, P-invariant function

$$q : \mathcal{P} \to P_+. \tag{2.58}$$

Now, a function q as in (2.58) is evidently equivalent to a P-invariant trivialization of the P_+-PFB $\pi_+ : \mathcal{P} \to \mathcal{G}_0$. Since an algebraic Weyl structure is fixed, this identifies G_0 as a subgroup of P. In particular, we have the following original definition of a Weyl structure from [22]:

Definition 26 *Given a fixed algebraic Weyl structure for a parabolic pair* (G, P)*, a* Weyl structure *for a parabolic geometry* $(\mathcal{P}, \pi, M, \omega)$ *of type* (G, P) *is a* G_0*-equivariant section* σ *of the projection*

$$\pi_+ : \mathcal{P} \to \mathcal{G}_0.$$

Proposition 27 *Given a fixed algebraic Weyl structure, there is a natural bijection between the Weyl structures of Definition 25 and those of Definition 26. Moreover, a Weyl structure is equivalently determined by any of the information given in (2.55), (2.56) or (2.57).*

Proof: The equivalence of Definitions 25 and 26 is given in Proposition A.1 of [12]. The equivalence of Definition 25 and the object (2.55) follows from the corresponding equivalence at the algebraic level, cf. Remark 3.3 of [12]. On the other hand, it's clear that an equivariant isomorphism (2.56) determines a grading section of $\mathcal{A}^0(M)$, as the first component of $\mathcal{E}_\bullet$, which maps $\mathcal{P}$ onto itself, may be taken by equivariance to be the identity. Finally, such an equivariant isomorphism is equivalent to a bundle isomorphism as in (2.57). □

The different expressions for a Weyl structure are useful for different purposes, and since we always will work with fixed algebraic Weyl structures, we shall use them interchangeably. Now we want to introduce some of the properties and geometry of Weyl structures, in terms of Definition 26, following [22]. First the existence is clarified (cf. Proposition 3.2 of [22]):

Proposition 28 *Weyl structures exist for any parabolic geometry* $(\mathcal{P}, \pi, M, \omega)$*. Moreover, if* σ *and* $\hat{\sigma}$ *are two Weyl structures, then there exists a unique smooth section* $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_k)$ *of*

$$\mathrm{gr}_1\mathcal{A}(M) \oplus \ldots \oplus \mathrm{gr}_k\mathcal{A}(M) = \mathrm{gr}(T^*M)$$

such that

$$\hat{\sigma}(u) = \sigma(u)\exp(\Upsilon_1(u))\ldots\exp(\Upsilon_k(u)). \tag{2.59}$$

Finally, each Weyl structure σ *and section* Υ *define another Weyl structure via (2.59).*

A Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{P}$ gives a reduction of the P-PFB $\mathcal{P}$ to the structure group G_0. In particular, the pullback $\sigma^*\omega$ of the Cartan connection is a G_0-equivariant form

$$\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g}). \tag{2.60}$$

Since G_0 preserves the grading of $\mathfrak{g}$, we can decompose this and get, for each $-k \leq i \leq k$, a well-defined, G_0-equivariant one-form

$$\sigma^*\omega_i \in \Omega^1(\mathcal{G}_0; \mathfrak{g}_i).$$

Proposition 29 *For a Weyl structure* σ*, define* $\theta^\sigma := \sigma^*\omega_-$*,* $\omega^\sigma := \sigma^*\omega_0$ *and* $\mathrm{P}^\sigma := \sigma^*\omega_+$*. Then* ω^σ *defines a* G_0 *principal bundle connection on* $\mathcal{G}_0$*, while* θ^σ *and* P^σ *are both horizontal and* G_0*-equivariant, and thus induce* the soldering one-form

$$\theta^\sigma \in \Omega^1(M; \mathrm{gr}_{-k}\mathcal{A}(M) \oplus \ldots \oplus \mathrm{gr}_{-1}\mathcal{A}(M)), \tag{2.61}$$

which induces an isomorphism

$$TM \cong \mathrm{gr}_{-k}\mathcal{A}(M) \oplus \ldots \oplus \mathrm{gr}_{-1}\mathcal{A}(M) \cong \mathrm{gr}(TM); \tag{2.62}$$

and the Rho-tensor

$$\mathrm{P}^\sigma \in \Omega^1(M; \mathrm{gr}(T^*M)). \tag{2.63}$$

For these properties, which follow pretty directly from the definitions, see 3.3 and 3.4 of [22]. We note that, using the expression for a Weyl structure given in

(2.56), it is possible to give a P-invariant lift of the above decomposition of the Cartan connection, cf. Appendix A of [12]:

$$\mathcal{E}_{\bullet}^{-1} \circ \omega = \omega_{\mathfrak{g}_-} + \omega_{\mathfrak{g}_0} + \omega_{\mathfrak{g}_+} \tag{2.64}$$
$$= \pi^*\theta^\sigma + \pi_+^*\omega^\sigma + \pi^*\mathrm{P}^\sigma. \tag{2.65}$$

The prototypical example of a Weyl structure on a parabolic geometry, is given by a choice of a particular semi-Riemannian metric g in a conformal class, as is described in Chapter 3.1. Given one such fixed metric, clearly any other metric $\tilde{g}$ in the conformal class is determined by a scale, i.e. $\tilde{g} = e^{2\phi}g$ for some $\phi \in C^\infty(M)$. This notion of scales can in fact be extended to Weyl structures for all parabolic geometries:

Definition 30 *An element $\varepsilon_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ is called a* scaling element *if and only if ε_λ acts by a nonzero real scalar on each G_0-irreducible component of $\mathfrak{g}_+$. A* bundle of scales *is a principal $\mathbb{R}^+$ bundle*

$$\pi_\lambda : \mathcal{L}^\lambda \to M$$

which is associated to $\mathcal{G}_0$ via a homomorphism $\lambda : G_0 \to \mathbb{R}^+$, whose derivative is given by $\lambda'(A) = B_{\mathfrak{g}}(\varepsilon_\lambda, A)$ for some scaling element $\varepsilon_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$.
Given a choice of a bundle of scales $\mathcal{L}^\lambda$, a (local) scale *on M is a (local) smooth section of $\mathcal{L}^\lambda$.*

Proposition 31 *Let G be a semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading. Then the following holds:*
1. There are scaling elements in $\mathfrak{z}(\mathfrak{g}_0)$.
2. Any scaling element $\varepsilon_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$ gives rise to a canonical bundle $\mathcal{L}^\lambda$ of scales over each manifold endowed with a parabolic geometry of the given type.
3. Any bundle of scales admits global smooth sections, i.e. there always exist global scales.

For the proof, see 3.7 of [22]. In particular, the first statement follows since the grading element $\varepsilon_0 \in \mathfrak{z}(\mathfrak{g}_0)$ can be taken as a scaling element. Important properties of bundles of scales for Weyl structures are summarized in the following Lemma (cf. Lemma 3.8 of [22]):

Lemma 32 *Let $\sigma : \mathcal{G}_0 \to \mathcal{P}$ be a Weyl structure for a parabolic geometry $(\mathcal{P}, \pi, M, \omega)$ and let $\mathcal{L}^\lambda$ be a bundle of scales. Then:*
1. The Weyl connection $\omega^\sigma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ induces a principal bundle connection on the bundle of scales $\mathcal{L}^\lambda$.
2. $\mathcal{L}^\lambda$ is naturally identified with $\mathcal{G}_0/\ker(\lambda)$, the orbit space of the free right action of the normal subgroup $\ker(\lambda) \subset G_0$ on $\mathcal{G}_0$.
3. The form $\lambda' \circ \omega^\sigma \in \Omega^1(\mathcal{G}_0)$ descends to the connection form of the induced principal bundle connection on $\mathcal{L}^\lambda = \mathcal{G}_0/\ker(\lambda)$.
4. The composition of λ' with the curvature form of ω^σ descends to the curvature of the induced connection on $\mathcal{L}^\lambda$.

A fundamental feature of Weyl structures for parabolic geometries, is that they can be determined completely in terms of objects on a bundle of scales. In particular, the principal bundle connection given by the first statement of Lemma 32 completely determines the Weyl structure, and this moreover leads to a description of the Cartan bundle of the parabolic geometry in terms of scales. In order to state this result, which is Theorem 3.12 of [22], we first need a general fact, cf. 17.4 of [39]:

Lemma 33 *Let $p : E \to M$ be any principal fiber bundle. Then there exists a bundle $qp : QE \to M$ whose sections are exactly the principal bundle connections on E.*

Theorem 34 *Let $(\mathcal{P}, \pi, M, \omega)$ be a parabolic geometry and let $\mathcal{L}^\lambda$ be a bundle of scales. Then:*
1. The induced principal bundle connection on $\mathcal{L}^\lambda$ from statement (1) of Lemma 32 defines a bijective correspondence between the set of Weyl structures and the set of principal connections on $\mathcal{L}^\lambda$.
2. There is a canonical isomorphism

$$\mathcal{P} \cong \pi_0^* Q\mathcal{L}^\lambda$$

where $\pi_0 : \mathcal{G}_0 \to M$ is the projection map. Under this isomorphism, the choice of a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{P}$ is the pullback of the principal bundle connection on the bundle of scales $\mathcal{L}^\lambda$, viewed as a section of the bundle

$$q\pi_\lambda : Q\mathcal{L}^\lambda \to M.$$

Moreover, the principal action of G_0 is the canonical action on $\pi_0^* Q\mathcal{L}^\lambda$ induced from the action on $\mathcal{G}_0$, while the action of P_+ is given by:

$$\hat{\nabla}_\xi s = \nabla_\xi s + \sum_{\|\underline{j}\|+l=0} \frac{(-1)^{\underline{j}}}{\underline{j}!} (ad(\Upsilon_k)^{j_k} \circ \ldots \circ ad(\Upsilon_1)^{j_1}(\xi_l)) \bullet s, \tag{2.66}$$

for ∇ a connection on the canonical line bundle L^λ associated to $\mathcal{L}^\lambda$, s a section of this line bundle, $(\Upsilon_1, \ldots, \Upsilon_k) \in \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, $\underline{j} = (j_1, \ldots, j_k)$ a multi-index, and $\xi \sim (\xi_1, \ldots, \xi_k) \in \mathrm{gr}_{-1}\mathcal{A}(M) \oplus \ldots \oplus \mathrm{gr}_{-k}\mathcal{A}(M)$ determined by the soldering form, and $\bullet$ denotes the action of $\mathfrak{g}_0$ on the line bundle.

In particular, given this result it makes sense to distinguish certain Weyl structures by the properties of the induced principal bundle connection:

Definition 35 *Let a bundle $\mathcal{L}^\lambda$ of scales be fixed for a parabolic geometry $(\mathcal{P}, \pi, M, \omega)$. A Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{P}$ is* closed *if the induced principal bundle connection on $\mathcal{L}^\lambda$ (or equivalently the induced linear connection ∇ on L^λ) is flat.*

A Weyl structure is exact *if its associated principal bundle connection on $\mathcal{L}^\lambda$ arises from a global smooth section of $\mathcal{L}^\lambda$.*

These sub-classes are well-defined, as a result of Theorem 34, and it's also clear that every exact Weyl structure is automatically closed, since a principal bundle connection arising from a global smooth section automatically has trivial holonomy and is thus flat. The space of closed Weyl structures is an affine space modeled on the space of closed one-forms on M, while the exact Weyl structures form an affine space modeled on the exact one-forms on M, cf. 3.13 of [22].

Finally, the holonomy of the Weyl connection of an exact Weyl structure is automatically contained in $\ker(\lambda) \subset G_0$. For example, in (oriented) conformal geometry, the exact Weyl structures are exactly those corresponding to a globally defined metric in the conformal class, whose Weyl connection automatically has holonomy contained in $SO(p,q) \cong \ker(\lambda)$. For closed Weyl structures, one has analogous local properties.

2.5 Harmonic curvature

For our later results on special conformal holonomy (especially in Chapter 4.2), we'll need one more tool from the theory of general parabolic geometries. Harmonic curvature allows us to use facts about the Lie algebra cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$ and its irreducible components, which are computable using Kostant's generalization of the Bott-Borel-Weil Theorem (BBW), to in many cases conclude that certain components of the Cartan curvature vanish.

Let $(\mathcal{P}, \pi, M, \omega)$ be a regular and normal parabolic geometry of type (G, P), with curvature function κ. Then we have the *Bianchi identity* (cf. Proposition 4.9 of [21]):

Proposition 36 *The curvature function κ satisfies the equation*

$$(\partial \circ \kappa)(X, Y, Z) + \sum_{cycl} (\kappa(\kappa_-(X, Y), Z) + \tilde{X}\kappa(Y, Z)) = 0 \qquad (2.67)$$

for all $X, Y, X \in \mathfrak{g}_-$, where ∂ is the Lie algebra derivative, $\sum_{cycl}$ denotes the sum over cyclic permutations of (X, Y, Z), and $\tilde{X} = \omega^{-1}(X) \in \Gamma(T\mathcal{P})$.

A corollary of the Bianchi identity, proved by splitting the equation (2.67) into homogeneous parts, is (cf. Corollary 4.10 of [21]):

Corollary 37 *For $(\mathcal{P}, \pi, M, \omega)$ and κ as above, let $\kappa = \sum_{l=1}^{3k} \kappa^{(l)}$ be the splitting of the curvature into homogeneous components. Then $\partial \circ \kappa^{(1)}$ is identically zero. More generally, if $\kappa^{(j)}$ is identically zero for all $j < i$, then $\partial \circ \kappa^{(i)}$ is identically zero.*

Thus, since also $\partial^* \circ \kappa^{(l)} = 0$ by normality, the isomorphism (2.28) allows us to identify $\kappa(u)$ with an element of the second cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g}) \cong Ker(\Box)$, for all $u \in \mathcal{P}$. In general, we define:

Definition 38 *The harmonic curvature function κ_H is the image of κ under the map*

$$\ker(\partial^*) \to \ker(\partial^*)/\mathrm{im}(\partial^*) \cong H^2(\mathfrak{g}_-, \mathfrak{g}).$$

The harmonic curvature is a much easier object to deal with than the full curvature, and yet it often provides quite a bit of information. For example, in conformal geometry the harmonic curvature of the canonical Cartan connection just corresponds to the (classical) Weyl curvature tensor, cf. Chapter 3.1. In general, as a result of Kostant's Theorem (see below), the second cohomology group may be viewed either as a P-module with trivial P_+-action, or a G_0-module. Thus, we may form the associated natural vector bundle, which can be viewed as associated to either $\mathcal{P}$ or $\mathcal{G}_0$:

$$\begin{aligned} \mathcal{H}^2(M) &:= \mathcal{P} \times_{(P,\bar{\rho})} H^2(\mathfrak{g}_-, \mathfrak{g}) \\ &\cong \mathcal{G} \times_{(G_0,\bar{\rho})} H^2(\mathfrak{g}_-, \mathfrak{g}), \end{aligned}$$

and $\kappa_H \in \Gamma(\mathcal{H}^2(M))$.

The following (cf. Proposition 4.12 of [21]) generalizes the classical result for conformal geometry, that vanishing of the Weyl curvature is a sharp obstruction for local conformal flatness:

Proposition 39 *Let $(\mathcal{P}, \pi, M, \omega)$ be a normal, regular parabolic geometry of type (G, P). Then the following are equivalent:*
1. The Cartan bundle is flat.
2. The harmonic curvature function κ_H vanishes.
3. The mapping $\mathfrak{g} \to \Gamma(T\mathcal{P})$ given by $X \mapsto \tilde{X}$ is a homomorphism of Lie algebras.
4. M is locally isomorphic to G/P.

To get restrictions on the harmonic curvature, we need to use Kostant's generalization of BBW to compute the irreducible components of $H^2(\mathfrak{g}_-, \mathfrak{g})$. For a useful guide to computing Lie algebra cohomology, using Kostant's result and the approach via Dynkin diagrams due to [3], see [57] (for the complex case) and [58] (for real cohomologies). Alternatively, an online version of the algorithm due to the same author can be found at `www.math.muni.cz/~silhan/lac`. Using this, the relevant cohomology components (and many others) for lower dimensions can be quickly determined by inputting the Dynkin/Satake diagram description of the infinitesimal parabolic pair. We present here the basic ingredients for the statement of Kostant's result and the application using these algorithms.

Consider, as for Definition 13, a Cartan algebra $\mathfrak{c}$ for $\mathfrak{g}$ and a choice of simple, positive roots $\Delta^0 \subset \Delta^+ \subset \Delta(\mathfrak{c}, \mathfrak{g})$ for this Cartan subalgebra, and let $\Sigma \subseteq \Delta^0$ be a subset corresponding to the standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. By definition, irreducible representations of $\mathfrak{p}$ are irreducible representations of $\mathfrak{g}_0$ with the trivial action of $\mathfrak{p}_+$. Weights of $\mathfrak{p}$ can be represented with labeled Dynkin/Satake diagrams of $\mathfrak{g}$, where all coefficients over non-crossed nodes are integers. A weight is dominant for $\mathfrak{p}$ (and thus corresponds to an irreducible $\mathfrak{p}$-representation) if and only if the coefficients over non-crossed nodes are all non-negative. Then we have:

Definition 40 *The* Weyl group *W is the group generated by* simple reflections*, i.e. reflections corresponding to simple roots $\alpha \in \Delta^0$.*
For an element $w \in W$, the number of positive roots $\alpha \in \Delta^+$ which are transformed to $w(\alpha) \in \Delta^- := -\Delta^+$ is called the length *of w.*
The subset $W^{\mathfrak{p}} \subseteq W$ consists of all elements of the Weyl group which map weights dominant for $\mathfrak{g}$ into weights dominant for $\mathfrak{p}$. Equivalently, $W^{\mathfrak{p}}$ is the set

$$W^{\mathfrak{p}} = \{w \in W | \Phi_w := w(\Delta^-) \cap \Delta^+ \subseteq \Delta(\mathfrak{p}_+)\}.$$

Now, given a representation $\nu : \mathfrak{g} \to \mathfrak{gl}(V)$ and its restriction $\pi = \nu_{|\mathfrak{p}_+}$, we get a natural representation

$$\beta : \mathfrak{p} \to \mathfrak{gl}(Hom(\Lambda^n \mathfrak{p}_+, V)).$$

This factorizes to a representation

$$\bar{\beta} : \mathfrak{p} \to \mathfrak{gl}(H^n(\mathfrak{p}_+, V)),$$

where $H^n(\mathfrak{p}_+, V)$ are the standard cohomology groups for the chain complex with chain groups $C^n(\mathfrak{p}_+, V) := Hom(\Lambda^n \mathfrak{p}_+, V)$. Part of Kostant's result is that this representation is completely reducible, and thus we may consider just its restriction

$$\bar{\beta} : \mathfrak{g}_0 \to \mathfrak{gl}(H^n(\mathfrak{p}_+, V)).$$

Theorem 41 ([40], Kostant's generalization of BBW) *For the finite dimensional representation $\nu : \mathfrak{g} \to \mathfrak{gl}(V)$ with the highest weight λ and the restriction $\pi = \nu_{|\mathfrak{p}}$, the irreducible components of $\bar{\beta}$ are in bijective correspondence with the set $W^{\mathfrak{p}}$ and the multiplicity of each component is one. The highest weight of the irreducible component of the representation $\bar{\beta}$ corresponding to $w \in W^{\mathfrak{p}}$ is $w.\lambda = w(\lambda + \rho) - \rho$ and it occurs in degree $|w|$. The generator of this component (the vector of the highest weight) is $\Lambda_{\alpha \in \Phi_w} \mathfrak{g}_\alpha \to s_{w\lambda}$, where $s_{w\lambda} \in V$ is a weight vector of the weight $w\lambda$.*

For our purposes, i.e. for computing $H^2(\mathfrak{g}_-, \mathfrak{g})$, we use the representation $\mathrm{ad}_{|\mathfrak{g}_-} : \mathfrak{g}_- \to \mathfrak{g}$. Since $\mathfrak{g}_- \cong (\mathfrak{p}_+)^*$ and the adjoint representation is self-dual, we have $H^2(\mathfrak{g}_-, \mathfrak{g}) \cong (H^2(\mathfrak{p}_+, \mathfrak{g}))^*$. The relevant information on this cohomology group is given as we deal with each case (cf. Chapters 3.1, 4.1, respectively).

Finally, we cite a result generalizing Proposition 39, which allows us, even in cases where κ_H as a whole is non-vanishing, to draw conclusions about the values of the curvature function κ based on the vanishing of certain components of κ_H. Together with the algorithm for computing the irreducible components of $H^2(\mathfrak{g}_-, \mathfrak{g})$, this gives very useful tools, which we'll make use of especially in the computations of normality in Chapter 4.2. As opposed to Proposition 39, which follows rather easily from the Bianchi identity, the proof of the following Proposition (cf. Corollary 3.2 of [13]) requires techniques of curved BGG sequences (cf. [11] or [24]) which are beyond the scope of this work.

Proposition 42 *Let $\mathbb{E} \subset \ker(\partial^*) \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ be a P-module, and set $\mathbb{E}_0 := \mathbb{E} \cap \ker(\Box)$. If either the Cartan connection is torsion-free, or $\Box(\mathbb{E}) \subset \mathbb{E}$ and $\mathbb{E}$ is stable under $\mathbb{E}$-insertions, then $\kappa_H \in \mathbb{E}_0$ implies that also $\kappa \in \mathbb{E}$.*

The definition of *stability under $\mathbb{E}$-insertions* for a P-module is rather technical, cf. Definition 3.2 of [13]. But it is easy to verify that the module $\Lambda^2(g_-, \mathfrak{p})$ satisfies the condition, and this is the only case which we'll need in practice. In other words, Proposition 42 implies, in particular, that if the harmonic curvature is torsion-free, then the full curvature is also torsion-free.

Chapter 3

Conformal holonomy

3.1 Conformal geometry as a parabolic geometry

The starting point for describing conformal geometry as a parabolic geometry is the homogeneous model space G/P. This is the Möbius sphere $(S^{p,q}, c)$ with the standard conformal structure, which in Riemannian signature is just the usual n-sphere with the conformal structure induced by the standard metric.

To describe this conformal manifold as a homogeneous space, consider first the pseudo-Euclidean space of signature $(p+1, q+1)$

$$(\mathbb{R}^{p+1,q+1}, <,>_{p+1,q+1}),$$

with a basis $\{e_0, \ldots, e_{p+q+1}\}$ such that the metric $<,>_{p+1,q+1}$ is given by the quadratic form

$$I^{1,1}_{p,q} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

where $I_{p,q}$ is the standard quadratic form of signature (p, q).

Let $C^{p,q} \subset \mathbb{R}^{p+1,q+1}$ be the *real light cone*, i.e. the hypersurface consisting of those non-zero vectors which are null with respect to $<,>_{p+1,q+1}$, and let

$$p : C^{p,q} \to \mathbb{RP}(C^{p,q}) =: S^{p,q}$$

be the real projectivization map. The *conformal Möbius sphere* is this space together with the conformal structure induced as follows: Choose a smooth section $\sigma \in C^\infty(S^{p,q}, C^{p,q}_0)$ of the projectivization map with values in a fixed connected component of the light cone, and let $g^\sigma := \sigma^* <,>_{p+1,q+1}$.

This defines a metric of signature (p, q) on $S^{p,q}$, and any other such section differs by a positive-valued smooth function, inducing another metric in the conformal class. It is a classical result that the automorphism group of this conformal manifold is $G := PSO(p+1, q+1)$ (the quotient of $SO(p+1, q+1)$ by its center), and that it is locally conformally flat. Thus, G/P is a homogeneous model space for conformal geometry of signature (p, q), if we let P be the stabilizer of any real isotropic (i.e. null) line in $\mathbb{R}^{p+1,q+1}$.

To see the parabolic structure of conformal geometry, we give an explicit grading of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$. For this, take $P = \mathrm{stab}_G(L_\mathbb{R})$, for $L_\mathbb{R} := \mathbb{R}e_0$

the real span of the first standard basis vector of $\mathbb{R}^{p+1,q+1}$ (the form of $I^{1,1}_{p,q}$ is chosen so that this, "extra" basis vector will be null). Then it is straightforward to see that

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$$

gives a $|1|$-grading of $\mathfrak{g}$ corresponding to the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$, where:

$$\mathfrak{g}_{-1} = \{\mathfrak{g}_{-1}(X) := \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^{\psi t} & 0 \end{pmatrix} \mid X \in \mathbb{R}^{p,q}\}; \tag{3.2}$$

$$\mathfrak{g}_0 = \{\mathfrak{g}_0(a, A) := \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix} \mid a \in \mathbb{R}, A \in \mathfrak{so}(p,q)\}; \tag{3.3}$$

$$\mathfrak{g}_{+1} = \{\mathfrak{g}_{+1}(Z) := \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & -Z^{\psi t} \\ 0 & 0 & 0 \end{pmatrix} \mid Z \in (\mathbb{R}^{p,q})^*\}. \tag{3.4}$$

We write here, e.g. $X^{\psi t}$ for the *pseudo-transpose* of X: $X^{\psi t} := (I_{p,q}X)^t$.

In particular, the maps $\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_+$ defined in (3.2), (3.3) and (3.4), respectively, give an isomorphism

$$\mathfrak{g} \cong \mathbb{R}^{p,q} \oplus \mathfrak{co}(p,q) \oplus (\mathbb{R}^{p,q})^*.$$

So a regular infinitesimal flag structure of type $(\mathfrak{g}, \mathfrak{p})$ on a manifold M of dimension $n = p + q$, is just the vacuous filtration on TM (regularity is also vacuous, as is always the case for $|1|$-gradings), together with a reduction of the frame bundle $GL(M)$ to a structure group G_0 with Lie algebra $\mathfrak{g}_0 \cong \mathfrak{co}(p,q)$. This is nothing but a conformal structure on M.

Note that there is a choice here of what group to take, from non-oriented conformal structures, to time- and/or space-oriented conformal structures, or even conformal spin structures. These correspond to choosing different groups G_0, which also effects the group G, as was mentioned at the end of Chapter 2.3. We see here how different topological obstructions – vanishing of certain Whitney classes, etc. – come into play if one asks for a particular group G. We will generally assume maximal orientability, i.e. the existence of a reduction of the frame bundle to $CO_0(p,q) = \mathbb{R}^+ \rtimes SO_0(p,q)$. Then we can take G to be $SO(p+1, q+1)$, and for holonomy theory (where it's practical to restrict to connected conformal holonomy groups) we will even have a reduction to the orthogonal group preserving fixed space and time orientations $SO_0(p+1, q+1)$.

From Theorem 23 we have for a conformal manifold (M, c) of signature (p,q), a normal Cartan geometry $(\mathcal{P}, \pi, M, \omega)$ of type (G, P) as above, which is (up to isomorphism) canonical. As mentioned in Chapter 2.4, a choice of metric in the conformal class determines a (exact) Weyl structure for the parabolic geometry: Given $g \in c$, we have a principal connection on the conformal frame bundle $\mathcal{G}_0$, namely that induced by the Levi-Civita connection for g, $\omega^{LC}(g)$. This induces a principal bundle connection on any bundle of scales, which via pull-back by π_0 as explained in Chapter 2.4 induces a Weyl structure $\sigma(g) : \mathcal{G}_0 \to \mathcal{P}$. Since $Hol(\omega^g) \subseteq SO_0(p,q)$, the induced connection on the bundle of scales must come from a global trivialization, so $\sigma(g)$ is exact.

Using a choice of metric $g \in c$ and its induced Weyl structure, in fact we can describe the normal Cartan connection for (M, c) in terms of geometric quantities of g. The derivation of the following formulae from the point of view of the prolongation construction of a canonical Cartan geometry in the conformal setting, is discussed in section 3.3 of [4] with reference to [24]:

Proposition 43 *Let (M, c) be a maximally oriented conformal manifold of signature (p, q) and let $(\mathcal{P}, \pi, M, \omega)$ be the normal parabolic geometry of type (G, P) determined by it. Then for a metric $g \in c$ and the associated Weyl structure $\sigma(g)$, we have:*

$$\theta^{\sigma(g)} = \theta^{[g]}; \tag{3.5}$$

$$\omega^{\sigma(g)} = \omega^{LC}(g); \tag{3.6}$$

$$\mathrm{P}^{\sigma(g)} = \mathrm{P}^{g}. \tag{3.7}$$

Here, P^g *denotes the classical* Schouten- *or* Rho-tensor *of conformal geometry:*

$$\mathrm{P}^g = \frac{1}{n-2}\left(\frac{1}{2(n-1)}R^g g - Ric^g\right).$$

The most important Tractor bundles for conformal geometry are the adjoint bundle $\mathcal{A}(M)$ as introduced above for general parabolic geometries, and the standard Tractor bundle: Let $\rho : G = SO(p+1, q+1) \to GL(\mathbb{R}^{p+q+2})$ be the standard representation, then the standard Tractor bundle in conformal geometry is:

$$\mathcal{T}(M) := \mathcal{P} \times_{(\rho,P)} \mathbb{R}^{p+q+2}. \tag{3.8}$$

Since $P \subset G$ preserves the metric $<,>_{p+1,q+1}$, the vector bundle $\mathcal{T}(M)$ inherits a metric of signature $(p+1, q+1)$, denoted by $<,>_{\mathcal{T}}$. It also follows this metric is parallel with respect to the Tractor connection $\nabla^{\mathcal{T}}$, defined via the principal Tractor bundle $(\mathcal{G}, \bar{\pi}, M, \bar{\omega})$.

Furthermore, $\mathcal{T}(M)$ inherits a filtration, from the P-invariant filtration of $\mathbb{R}^{p+q+2}$:

$$\mathbb{R}^{p+q+2} \supset (L_{\mathbb{R}})^{\perp} \supset L_{\mathbb{R}} \supset \{0\}.$$

The filtration on $\mathcal{T}(M)$ is given by:

$$\begin{aligned}\mathcal{T}(M) &= \mathcal{T}^{-1}(M) \supset \mathcal{T}^{0}(M) \supset \mathcal{T}^{1}(M)\\ &:= (\mathcal{P} \times_{(\rho,P)} \mathbb{R}^{p+q+2}) \supset (\mathcal{P} \times_{(\rho,P)} (L_{\mathbb{R}})^{\perp}) \supset (\mathcal{P} \times_{(\rho,P)} L_{\mathbb{R}}).\end{aligned}$$

In particular, the line bundle $\mathcal{T}^1(M)$ is isotropic with respect to $<,>_{\mathcal{T}}$. A non-vanishing, global smooth section $\alpha \in \Gamma(\mathcal{T}^1(M))$ induces a metric g on M of signature (p, q). For the quotient $\mathcal{T}^0(M)/\mathcal{T}^1(M)$ we have an isomorphism of associated G_0 bundles

$$\mathcal{T}^0(M)/\mathcal{T}^1(M) \cong TM[-1],$$

where $TM[-1] = TM \otimes \mathcal{E}[-1]$ is the conformally weighted tangent bundle, and $\mathcal{E}[-1]$ is a certain density bundle, which can be naturally identified with $\mathcal{T}^1(M)$, cf. section 2 of [2]. Then the section α determines an identification of $TM[-1]$ with a sub-bundle of $\mathcal{T}^0(M)$, and the restriction of $<,>_{\mathcal{T}}$ to this sub-bundle is non-degenerate, determining a metric on M. Choosing another non-vanishing global section $\alpha' = e^{2\phi}\alpha \in \mathcal{T}^1(M)$ gives a metric which differs by a conformal factor. From this description it is clear how the Cartan geometry (or its standard associated Tractor bundle) provides a curved version of the homogeneous conformal structure described at the outset for $S^{p,q}$.

On the other hand, a choice of metric $g \in c$ and the associated Weyl structure $\sigma(g)$, give a reduction $\sigma(g)' : \mathcal{SO}_0(M,g) \to \mathcal{P}$ to the structure group $SO_0(p,q)$. This induces a splitting of the standard Tractor bundle:

$$\begin{aligned}\mathcal{T}(M) &= \mathcal{P} \times_{(P,\rho)} \mathbb{R}^{p+q+2} \\ &\overset{\sigma(g)'}{\cong} \mathcal{SO}_0(M,g) \times_{(SO_0(p,q),\rho)} (\mathbb{R} \oplus \mathbb{R}^{p+q} \oplus \mathbb{R}) \\ &\overset{g}{\cong} (M \times \mathbb{R}) \oplus TM \oplus (M \times \mathbb{R}).\end{aligned}$$

The following result gives the form of $\nabla^{\mathcal{T}}$ and its curvature endomorphism $\mathcal{R}^{\mathcal{T}}$ in terms of such a splitting. From the description of the normal conformal Cartan connection in terms of an exact Weyl structure (i.e. a metric in the conformal class), it is a straightfroward computation using the standard representation of $SO(p+1,q+1)$, cf. Section 3 of [44], or Proposition 10 of [4], and from this point of view the conformal invariance is direct. On the other hand, a direct construction of the Tractor bundle and connection was already given, based on the work of T. Thomas and in modernized form, in section 2 of [2]. This agrees with the normal connection, cf. [16].

Proposition 44 *Let g be a metric in the conformal class and*

$$\Gamma(\mathcal{T}(M)) \overset{g}{\cong} C^\infty(M) \oplus \Gamma(TM) \oplus C^\infty(M)$$

the corresponding splitting of sections of the standard Tractor bundle. With respect to this splitting, the Tractor connection on $\mathcal{T}(M)$ induced by the canonical Cartan connection ω of the conformal structure, has the matrix form:

$$\nabla^{\mathcal{T}}_X = \begin{pmatrix} \nabla^{LC}_X & \mathrm{P}^g(X) & 0 \\ X\cdot & \nabla^{LC}_X & -\mathrm{P}^g(X)^\sharp \cdot \\ 0 & -X^* & \nabla^{LC}_X \end{pmatrix}. \tag{3.9}$$

The curvature endomorphism of $\nabla^{\mathcal{T}}$ has the form:

$$\mathcal{R}^{\mathcal{T}}(X,Y) = \begin{pmatrix} 0 & C^g(X,Y)^* & 0 \\ 0 & W^g(X,Y)\bullet & -C^g(X,Y)\cdot \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

Here, C^g denotes the Cotton-York tensor, W^g the Weyl curvature tensor of g, and X, Y are vector fields on M.

In particular, from the matrix form (3.10) we can identify the homogeneity components of the curvature function κ of the canonical Cartan connection of a conformal manifold (M,c). The diagonal entries have values in $\mathfrak{g}_0$, while the upper-diagonal entries have values in $\mathfrak{g}_1$. Since $X_1, X_2 \in TM$ correspond to elements in $\mathfrak{g}_- = \mathfrak{g}_{-1}$, this means that W^g corresponds to $\kappa^{(2)}$ and C^g corresponds to $\kappa^{(3)}$. In particular, we see that the canonical Cartan connection of a conformal structure is always torsion-free.

In fact, this last fact can be determined from the cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$ using the techniques described in Chapter 2.5. The Satake diagram for the parabolic pair $(\mathfrak{g}, \mathfrak{p})$ for conformal structures (described above) is the usual Satake diagram for the non-compact real form $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$ of either $B_{\frac{p+q+1}{2}}$ or $D_{\frac{p+q+2}{2}}$, where the first node of the diagram (which is non-compact) is crossed. Using the algorithm in [58] or visiting the same author's website, one sees that $H^2(\mathfrak{g}_-, \mathfrak{g})$ has exactly one irreducible component, which is of homogeneity 2 in dimensions higher than 3. On the other hand, in dimension 3 the Weyl curvature always vanishes and the Cotton-York tensor gives the whole curvature. Applying this, together with Corollary 37 and Proposition 39 gives:

Proposition 45 *Let (M, c) be a conformal manifold and $(\mathcal{P}, \pi, M, \omega)$ its canonical parabolic geometry, with curvature function κ. Then ω is torsion-free, and the harmonic curvature $\kappa_H = \kappa^{(2)}$ corresponds to the Weyl curvature tensor W. A semi-Riemannian manifold (M, g) of dimension n is locally conformally flat if and only if, respectively, W^g $(n \geq 4)$, or C^g $(n = 3)$ vanishes.*

3.2 Decomposable conformal holonomy

Definition 46 *Let $(\mathcal{P}, \pi, M, \omega)$ be a Cartan geometry of arbitrary type (G, P). The* holonomy group *of the Cartan geometry is the holonomy group $Hol(\bar{\omega})$ of its principal Tractor bundle $(\mathcal{G}, \bar{\pi}, M, \bar{\omega})$.*
For a manifold M endowed with a regular infinitesimal flag structure of type $(\mathfrak{g}, \mathfrak{p})$, the holonomy group of this geometric structure is the holonomy group of the (canonical) regular, normal parabolic geometry associated to it.
In particular, for a maximally oriented conformal manifold (M, c) of signature (p, q), the conformal holonomy group $Hol(M, c)$ is the holonomy group of the canonical parabolic geometry described in Chapter 3.1

We note that this definition of conformal holonomy differs slightly from that given in other places in the literature, defined via the development map for a Cartan geometry, cf. [56] or [4]. However, this should not lead to considerable confusion, given the following result (cf. Proposition 1 of [4]):

Proposition 47 *1. The connected component of the holonomy group given by Definition 46 is isomorphic to the connected component of the holonomy group of the Cartan geometry defined via the development map.*
2. If the structure group P is connected or the base space M is simply connected, then the full holonomy groups are isomorphic. Moreover, for any representation $\rho : G \to GL(W)$ and associated Tractor bundle $(\mathcal{W}, \nabla^{\mathcal{W}})$, we have an isomorphism

$$\rho(Hol(\bar{\omega})) \cong Hol(\nabla^{\mathcal{W}})$$

As mentioned in Chapter 3.1, we will always assume maximal orientability of the conformal structure, and in particular since we're mainly interested in connected conformal holonomy groups, this discrepancy in definitions is irrelevant. Proposition 47 also allows us to identify $Hol(M, c)$ with the matrix group $Hol(\nabla^{\mathcal{T}})$, the holonomy group of the standard Tractor connection given, via a choice of metric $g \in c$, by (3.9).

This last identification allows us to give the geometric meaning of decomposable conformal holonomy. First, consider a conformal manifold (M, c) with a one-dimensional $Hol(M, c)$-invariant subspace. Since $\mathcal{T}(M)$ has an indefinite metric, and $\nabla^{\mathcal{T}}$ respects this metric, we have as usual for linear, metric connections, a bijection between such holonomy-invariant subspaces and *recurrent Tractors*, i.e. non-vanishing sections $\Upsilon \in \mathcal{T}(M)$ such that

$$\nabla^{\mathcal{T}} \Upsilon = \gamma \Upsilon$$

for some one-form $\gamma \in \Omega^1(M)$. If a recurrent Tractor Υ has non-zero length at any point with respect to $<,>_{\mathcal{T}}$, then it is straightforward to see that it can be locally rescaled to a parallel section by dividing by the length. Furthermore, the standard conformal Tractor connection has the surprising property that *all* recurrent Tractors can locally be rescaled to parallel sections, cf. Lemma 2.1 of [43].

Thus, one-dimensional $Hol(M,c)$-invariant subspaces correspond to parallel standard Tractors $\Upsilon \in \mathcal{T}(M)$. Now choose a metric $g \in c$ and let

$$\Upsilon = \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix}$$

be the representation of the parallel Tractor Υ with respect to the splitting given by g, where $\alpha, \beta \in C^\infty(M)$ and Y is a vector field on M. From the formula (3.9), one sees that $\nabla^{\mathcal{T}}\Upsilon = 0$ implies

$$\begin{aligned} Y &= \mathrm{grad}^g \beta; \\ d\alpha &= -\mathrm{P}^g(\mathrm{grad}^g \beta); \\ \alpha g &= \beta \mathrm{P}^g - \mathrm{Hess}^g(\beta). \end{aligned}$$

Then the subset $\tilde{M}$ where β is non-vanishing must be dense (and open) in M, or else Υ would vanish identically. Consider the metric on $\tilde{M}$ given by $\tilde{g} := \beta^{-2} g \in c_{|\tilde{M}}$. A calculation using the conformal transformation rule for the Schouten tensor (cf. p. 57 of [5]) then shows that $\mathrm{P}^{\tilde{g}}$ is proportional to $\tilde{g}$, implying that $(\tilde{M}, \tilde{g})$ is Einstein.

Conversely, suppose $g \in c$ is an Einstein metric. Then

$$\mathrm{P}^g = -\frac{R^g}{2n(n-1)} g,$$

where R^g is the scalar curvature of g. Then defining

$$\Upsilon := \begin{pmatrix} -\frac{R^g}{2n(n-1)} \\ 0 \\ 1 \end{pmatrix}, \tag{3.11}$$

one sees from (3.9) that Υ is parallel. Furthermore, from the formula (3.11), we see that the sign of the length of the parallel Tractor is the opposite that of the scalar curvature of the Einstein metric to which it corresponds. Summarizing, we have:

Proposition 48 *For a conformal manifold (M,c), $Hol(M,c)$-invariant lines $\mathbb{R}v \subset \mathbb{R}^{p+1,q+1}$ invariant are in bijective correspondence to Einstein metrics $g \in c_{|\tilde{M}}$ defined up to singularity on M. Under this correspondence, we have:*

$$\begin{aligned} R^g > 0 &\Leftrightarrow < v, v > < 0; \\ R^g = 0 &\Leftrightarrow < v, v > = 0; \\ R^g < 0 &\Leftrightarrow < v, v > > 0. \end{aligned}$$

So Einstein metrics have a special place in the theory of reducible conformal holonomy. There is a decomposition theorem for conformal manifolds with so-called decomposable holonomy, due independently to F. Leitner [44] and S. Armstrong [1]. This can be seen both as a conformal analog of the de Rham/Wu Decomposition Theorem and a higher rank generalization of the above Proposition.

Definition 49 *A conformal manifold* (M, c) *is said to have* decomposable holonomy of rank k *if there exists a non-degenerate* $Hol(M, c)$*-invariant subspace* $V \subset \mathbb{R}^{p+1,q+1}$ *with* $1 < \mathrm{rk}(V) =: k < p + q + 1$*. Otherwise,* (M, c) *is said to have* indecomposable holonomy*.*

Proposition 50 (Conformal holonomy decomposition, Armstrong [1], Leitner [44]) *Let* (M, c) *be a conformal manifold with decomposable holonomy of rank* k*. Then for some open dense subset* $\tilde{M}$ *of* M*,* $(\tilde{M}, c)$ *is locally conformally isomorphic to a product* $(\tilde{M}_1^k \times \tilde{M}_2^{p+q-k+1}, [g_1 + g_2])$*, where* (M_i, g_i) *are Einstein manifolds whose scalar curvatures satisfy*

$$R^{g_1} = -\frac{k(k-1)}{(n-k)(n-k-1)} R^{g_2}.$$

Moreover, for a product manifold $(\tilde{M}, g) \cong (\tilde{M}_1 \times \tilde{M}_2, g_1 + g_2)$ *satisfying this relation, we have*

$$Hol(\tilde{M}, [g]) \cong Hol(\tilde{M}_1, [g_1]) \times Hol(\tilde{M}_2, [g_2]).$$

This decomposition result allows a more or less complete characterization of the possible non-irreducible conformal holonomy groups in some signatures. For a maximally oriented Riemannian conformal manifold (M^n, c), $Hol(M, c) \subseteq SO(1, n+1)$, and thus we can apply the following result (cf. Theorem 2 of [28]):

Theorem 51 *(Di Scala, Olmos [29]) The only connected, irreducible subgroup of* $O(1, n + 1)$ *is* $SO_0(1, n + 1)$*.*

Thus, if (M, c) is non-generic, $Hol(M, c)$ has an invariant subspace V. If V is degenerate, then $Hol(M, c)$ must also preserve a light-like line contained in V, and hence there is a Ricci-flat Einstein metric in the conformal class, defined up to singularities. If $1 < \mathrm{rk}(V) < n+1$, then Proposition 50 implies a local decomposition into conformally Einstein manifolds of smaller dimension. Thus in a sense, all non-generic conformal manifolds in Riemannian signature can be (locally) decomposed into indecomposable conformally Einstein manifolds. These were classified in [1] using an isomorphism between the standard Tractor bundle and a metric cone construction on M:

Proposition 52 *(Armstrong, [1]) Let* (M^n, c) *be a simply connected, conformally Einstein manifold of Riemannian signature with indecomposable conformal holonomy, and suppose* $n > 3$*. Then the possible conformal holonomy groups (and geometries of the unique Einstein manifold given by* $g \in c$*) are:*

1. $SO_0(1, n)$ *(generic,* $R^g < 0$*);*

2. $SO(n + 1)$ *(generic,* $R^g > 0$*);*

3. $SO(n) \rtimes \mathbb{R}^n$ *(generic,* $R^g = 0$*);*

4. $SU(\frac{n+1}{2})$ *(Einstein-Sasaki,* $R^g > 0$*);*

5. $SU(\frac{n}{2}) \rtimes \mathbb{R}^n$ *(Calabi-Yau,* $R^g = 0$*);*

6. $Sp(\frac{n+1}{4})$ *(3-Sasakian,* $R^g > 0$*);*

7. $Sp(\frac{n}{4}) \rtimes \mathbb{R}^n$ *(Hyper-Kähler,* $R^g = 0$*);*

8.$Spin(7) \rtimes \mathbb{R}^8$ *($Spin(7)$-holonomy, $R^g = 0$);*

9.$Spin(7)$ *if $n = 7$ (nearly G_2, $R^g > 0$);*

10.$G_2 \rtimes \mathbb{R}^7$ *(G_2-holonomy, $R^g = 0$);*

11.G_2 *if $n = 6$ (Nearly-Kähler, $R^g > 0$).*

Finally, we mention that there are also results, described in [43], characterizing the geometry of all Lorentzian conformal manifolds having non-irreducible holonomy. In Lorentzian signature, however, there is no classification result limiting the irreducible subgroups which could occur as conformal holonomy groups. Irreducible (and connected) conformal holonomy groups will be our focus in the sequel.

3.3 Transitive holonomy and Fefferman spaces

Connected, irreducible, transitive conformal holonomy groups have the benefit that we can give a nearly complete list of them (see Chapter 3.4). In the present section, we want to describe the geometric constructions (especially the generalized Fefferman construction) which link such subgroups to conformal holonomy. Note that throughout this and the following Chapters, we will always take (M, c) to be a conformal manifold of signature (p, q), $G := SO_0(p + 1, q + 1)$ and $P := \text{stab}_G(L_{\mathbb{R}})$ as described in Chapter 3.1. We also will sometimes need to consider $G' := SO(p + 1, q + 1)$ and $P' := \text{stab}_{G'}(L_{\mathbb{R}})$. In general, for any subgroup $H \subseteq SO(p + 1, q + 1)$, we also adopt the notation convention, $P^H := \text{stab}_H(L_{\mathbb{R}})$.

Definition 53 *A subgroup $H \subseteq G'$ is called* transitive *if H acts transitively on $S^{p,q} = G'/P'$.*
(M, c) has transitive holonomy *if $Hol(M, c) \subseteq G$ is transitive.*

Before describing the generalized Fefferman construction of conformal manifolds, we give a conformal holonomy reduction principle, which also shows why we need transitive subgroups:

Definition 54 *Let a transitive subgroup $H \subseteq SO(p + 1, q + 1)$ be given. A* Cartan reduction to H *of a parabolic geometry $(\mathcal{P}, \pi, M, \omega)$ of conformal type (G, P) is given by a Cartan geometry $(\mathcal{P}^H, \tilde{\pi}, M, \tilde{\omega})$ of type (H, P^H) together with an inclusion $\iota : \mathcal{P}^H \hookrightarrow \mathcal{P}$ such that $\iota^*\omega = \tilde{\omega}$.*

Proposition 55 *(Conformal Cartan reduction principle)*
Let $H \subset G'$ be a transitive subgroup and $(\mathcal{P}, \pi, M, \omega)$ be a parabolic geometry of conformal type (G', P'). There exists a Cartan reduction to H if and only if $Hol(\omega) \subseteq H$.

Proof: ($\Rightarrow$) Let $(\mathcal{P}^H, \tilde{\omega})$ be the Cartan reduction. Since $\iota : \mathcal{P}^H \hookrightarrow \mathcal{P}$ gives an inclusion, we also have an inclusion $\bar{\iota} : \mathcal{H} \hookrightarrow \mathcal{G}$, where $\mathcal{H} = \mathcal{P}^H \times_{P^H} H$ is the extension of $\mathcal{P}^H$ to a H-PFB. But there is a unique principal bundle connection $\bar{\tilde{\omega}}$ on $\mathcal{H}$ which pulls back under the natural inclusion to the Cartan connection $\tilde{\omega}$. Then since $\iota^*\omega = \tilde{\omega}$, applying H-equivariance one sees that $\bar{\iota}^*\bar{\omega} = \bar{\tilde{\omega}}$ By the holonomy reduction principle for principal bundle connections, it must hold that $Hol(\omega) := Hol(\bar{\omega}) \subseteq H$.

($\Leftarrow$) If $Hol(\omega) := Hol(\bar{\omega}) \subseteq H$, then the standard holonomy reduction principle gives a reduction to a H-PFB $(\mathcal{H}, \tilde{\bar{\omega}})$ of $(\mathcal{G}, \bar{\omega})$, i.e.

$$\bar{\iota} : \mathcal{H} \hookrightarrow \mathcal{G}$$

with $\bar{\iota}^* \bar{\omega} = \tilde{\bar{\omega}}$. Defining $\mathcal{P}^H := \mathcal{P} \cap \mathcal{H}$, $\mathcal{P}^H$ is clearly a principal fiber bundle over M with structure group P^H with a natural inclusion ι in $\mathcal{P}$. And $\tilde{\omega} := \tilde{\bar{\omega}}_{|T\mathcal{P}^H}$ is a one-form on $\mathcal{P}^H$ with values in $\mathfrak{h}$, and properties (2.1) and (2.2) of a Cartan connection are automatic, as is $\iota^* \omega = \tilde{\omega}$.

The final property (2.3) of a Cartan connection requires the transitivity of H, which implies in particular that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}. \tag{3.12}$$

The map

$$\tilde{\omega}_{|T_u \mathcal{P}^H} : T_u \mathcal{P}^H \to \mathfrak{h}$$

is injective for all $u \in \mathcal{P}^H$, since $\tilde{\omega} = \iota^* \omega$. Comparing dimensions, $dim(T_u \mathcal{P}^H) = dim(M) + dim(\mathfrak{p} \cap \mathfrak{h})$, while from property (2.3) for the canonical Cartan connection we have $dim(\mathfrak{g}) = dim(M) + dim(\mathfrak{p})$. Thus $dim(T_u \mathcal{P}^H) = dim(\mathfrak{h})$ if and only if $dim(\mathfrak{g}) = dim(\mathfrak{h} + \mathfrak{p})$, i.e. if and only if (3.12) holds. $\square$

As the proof shows, it would be enough to require the slightly weaker property (3.12) of $H \subseteq G$, which is called *local transitivity.* In particular, for any conformal manifold (M, c) with connected holonomy, of course $Hol(M, c) \subseteq G := SO_0(p+1, q+1)$, so the Proposition says that we may take as given a parabolic geometry of type (G, P), which we shall do from now on.

Proposition 55 indicates why generalized Fefferman constructions may be expected to play a role in special conformal holonomy. Namely, for a parabolic geometry of type (H, Q), where $H \subset G$ is a closed, (locally) transitive subgroup, and $Q \supseteq P^H$, the generalized Fefferman construction produces a manifold with conformal structure, together with an induced Cartan geometry of conformal type which naturally has a Cartan reduction to H. This gives a natural analog to C. Fefferman's construction, introduced in [31] for the smooth boundary of a pseudo-convex domain and subsequently generalized in [10] to arbitrary pseudo-convex CR manifolds.

It must be emphasized that this induced Cartan connection is not known in general to be ***normal***, even when the parabolic geometry of type (H, Q) is normal and regular. This must be shown in specific cases in order to prove that the conformal holonomy of the Fefferman space, which is by definition the holonomy of the normal Cartan connection, in fact reduces to H. Before getting into this for specific cases, we describe the general Fefferman construction due to A. Čap, cf. Section 4 of [14].

Let $(\mathcal{Q}, \pi', N, \omega')$ be a parabolic geometry of type (H, Q), where $H \subseteq G$ is transitive and $Q \supseteq P^H$. Then we can define the manifold

$$M := \mathcal{Q}/P^H.$$

In general, M fibers over N with fiber diffeomorphic to Q/P^H, which we'll denote

$$p : M \to N.$$

And by definition, $\mathcal{Q}$ is a P^H-PFB over M, which we'll denote by $\mathcal{P}^H$ when thought of this way:

$$\tilde{\pi} : \mathcal{P}^H := \mathcal{Q} \to M. \tag{3.13}$$

The Cartan connection ω' is a Cartan one-form on $\mathcal{P}^H = \mathcal{Q}$ with values in $\mathfrak{h}$, and it trivially satisfies the properties of a Cartan connection on $\mathcal{P}^H$ over M of type (H, P^H), and considered in this way we'll denote it $\tilde{\omega}$. Summarizing, we've defined an induced Cartan geometry $(\mathcal{P}^H, \tilde{\pi}, M, \tilde{\omega})$ of type (H, P^H).

Obviously, the notation for this induced Cartan geometry is chosen to remind us of the Cartan reductions of Definition 54. The notation is not misleading, and in fact we may associate to the Cartan geometry $(\mathcal{P}^H, \tilde{\pi}, M, \tilde{\omega})$ a parabolic Cartan geometry $(\mathcal{P}, \pi, M, \omega)$ of (maximally oriented) conformal type (G, P). This is defined as follows:

$$\mathcal{P} := \mathcal{P}^H \times_{P^H} P;$$

For $u \in \mathcal{P}^H \subset \mathcal{P}$, and for $\xi \in T_u\mathcal{P}$, write $\xi = \xi^H + \tilde{\xi_P}$, for $\xi^H \in T_u\mathcal{P}^H$ and $\xi_P \in \mathfrak{p}$, and define

$$\omega(\xi) := \tilde{\omega}(\xi^H) + \xi_P;$$

Finally, use *Ad*-equivariance to define ω over all points in $\mathcal{P}$. The properties (2.1) and (2.2) for a Cartan connection are automatically satisfied by ω, and the property (2.3) follows from the transitivity of H. We summarize the results of this construction in the following Proposition, which also gives the defining properties of the curvature functions κ^ω and $\kappa^{\tilde{\omega}}$, which are clear from the definitions of ω and $\tilde{\omega}$, respectively:

Proposition 56 *Given a parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of type (H, Q), where $Q \subseteq G$ is transitive and $Q \supseteq P^H$, there is an induced parabolic geometry $(\mathcal{P}, \pi, M, \omega)$ of conformal type (G, P), which induces a semi-Riemannian conformal metric c on M. This is the* (conformal) Fefferman space induced by $(\mathcal{Q}, \pi', N, \omega')$. *The conformal Fefferman space admits a natural Cartan reduction to H, which by Proposition 55 is equivalent to $Hol(\omega) \subseteq H$. Moreover, the curvature function κ^ω is characterized by the following commutative diagram, for all $u \in \mathcal{P}^H$, together with* Ad*-equivariance:*

$$\begin{array}{ccc}
\mathfrak{g}/\mathfrak{p} \wedge \mathfrak{g}/\mathfrak{p} & \xrightarrow{\kappa^\omega(u)} & \mathfrak{g} \\
\cong \downarrow & & \uparrow \\
\mathfrak{h}/\mathfrak{p}^H \wedge \mathfrak{h}/\mathfrak{p}^H & \xrightarrow{\kappa^{\tilde{\omega}}(u)} & \mathfrak{h} \\
\downarrow & & \uparrow = \\
\mathfrak{h}/\mathfrak{q} \wedge \mathfrak{h}/\mathfrak{q} & \xrightarrow{\kappa^{\omega'}(u)} & \mathfrak{h}
\end{array}$$

In other words, over the sub-bundle $\mathcal{P}^H$ the curvature function κ may be considered as having values either in the P-module $C^2(\mathfrak{g}_-, \mathfrak{g})$ or in the Q-module $C^2(\mathfrak{h}_-, \mathfrak{h})$. For calculating the normality in the next section, it is no problem to restrict to this sub-bundle, since $\ker(\partial^*) \subset C^2(\mathfrak{g}_-, \mathfrak{g})$ is Ad-invariant.

In particular, the curvature of the Cartan connection of the Fefferman space must vanish on the subspace $\mathfrak{q}/\mathfrak{p}^H$, i.e. $\kappa^{\tilde{\omega}}(X, Y) = 0$ for all $X \in \mathfrak{q}/\mathfrak{p}^H \subset \mathfrak{h}/\mathfrak{p}^H$ and all $Y \in \mathfrak{h}/\mathfrak{p}^H$. The final result of this Chapter shows that this condition is also sufficient for a local converse to the conformal Fefferman construction, given a Cartan reduction to H. This may be viewed as a generalization of G. Sparling's characterization of (classical) Fefferman spaces, cf. [35]:

Proposition 57 *Let $(\mathcal{P}, \pi, M, \omega)$ be a parabolic geometry of conformal type (G, P), suppose a Cartan reduction $(\mathcal{P}^H, \tilde{\pi}, M, \tilde{\omega})$ is given for a transitive, semi-simple subgroup $H \subseteq G$, and let $Q \supseteq P^H$ be a parabolic subgroup of H. Then $(\mathcal{P}, \pi, M, \omega)$ is locally isomorphic to the conformal Fefferman space induced by a parabolic geometry of type (H, Q) if and only if $\kappa^{\tilde{\omega}}$ vanishes on the subspace $\mathfrak{q}/\mathfrak{p}^H$.*

Proof: To begin with, the identifications

$$\begin{aligned} TM &\cong \mathcal{P} \times_P (\mathfrak{g}/\mathfrak{p}) \\ &\cong \mathcal{P}^H \times_{P^H} (\mathfrak{h}/\mathfrak{p}^H) \end{aligned}$$

induce a smooth distribution $\mathcal{V}^{\mathfrak{a}}$ for every subalgebra $\mathfrak{a} \supseteq \mathfrak{p}^H$ of $\mathfrak{h}$, defined by

$$\mathcal{V}^{\mathfrak{a}} = \mathcal{P}^H \times_{P^H} (\mathfrak{a}/\mathfrak{p}^H).$$

It is a straightforward result of the properties of the Cartan curvature, cf. Proposition 2.5 of [13], that $\mathcal{V}^{\mathfrak{a}}$ is integrable if and only if $\kappa^{\tilde{\omega}}$ preserves $\mathfrak{a}$, i.e. if and only if

$$\kappa^{\tilde{\omega}}(u)(\mathfrak{a}/\mathfrak{p}^H, \mathfrak{a}/\mathfrak{p}^H) \subseteq \mathfrak{a}.$$

In particular, given the assumptions of the proposition, we get an integrable distribution $\mathcal{V}^{\mathfrak{q}}$, and given any point $x \in M$ there exists a neighborhood U and a smooth submersion $\psi : U \to N$ onto the local leaf space of $\mathcal{V}^{\mathfrak{q}}$.

Now, local isomorphism to a conformal Fefferman space as in the Proposition may be formulated as follows: For $x \in M$ and $\psi : U \to N$ a local leaf space, let

$$\pi' : \mathcal{Q} := Q \times N \to N \tag{3.14}$$

be the trivial Q-PFB over the local leaf space. Then for every $u \in \mathcal{P}^H$, there must exist P^H-invariant open subsets $V \subset \mathcal{Q}$ and $U \subset \mathcal{P}^H$, and a P^H-equivariant diffeomorphism

$$\Phi : V \xrightarrow{\approx} U$$

such that $\psi \circ \tilde{\pi} \circ \Phi = \pi'$ and there exists a Cartan connection ω' on $\mathcal{Q}$ of type (H, Q) such that $\Phi^* \tilde{\omega} = \omega'$.

The proposition then follows from the proofs of Proposition 2.6 and Theorem 2.7 of [13] showing the existence of local twistor spaces for parabolic geometries under similar assumptions. In fact, the properties we require above of the local twistor space, do not depend in any essential way on the starting Cartan geometry being parabolic:

In the proof of Proposition 2.6 of [13], first suitable local diffeomorphisms Φ as above are constructed between P^H-invariant subsets. The key point is the existence of a local Q action on $\mathcal{P}^H$: For $A, B \in \mathfrak{q} \subset \mathfrak{h}$, consider the vector fields $\tilde{A}, \tilde{B}$ on $\mathcal{P}^H$ induced by $\tilde{\omega}$ as in (2.4). Then

$$\tilde{\omega}([\tilde{A}, \tilde{B}]) = -d\tilde{\omega}(\tilde{A}, \tilde{B}) = [A, B] - \kappa^{\tilde{\omega}}(A + \mathfrak{p}^H, B + \mathfrak{p}^H).$$

The assumption of the proposition on $\kappa^{\tilde{\omega}}$ implies in particular that the map $A \mapsto \tilde{A}$ gives a Lie algebra homomorphism of $\mathfrak{q}$ into the vector fields on $\mathcal{P}^H$. By Lie's second fundamental theorem, this infinitesimal action of $\mathfrak{q}$ on $\mathcal{P}^H$ integrates to a local group action of Q.

Using this local action and shrinking to sufficiently small neighborhoods U of $\mathcal{P}^H$ and $\tilde{V}$ of $e \in Q$ (for example, a smooth section of $\psi \circ \tilde{\pi} : U \to N$ must exist), Čap then defines a diffeomorphism

$$\Phi : \tilde{V} \times N \to U$$

such that $\psi \circ \tilde{\pi} \circ \Phi = \pi'$, and satisfying

$$T\Phi \circ \tilde{A} = \tilde{A} \circ \Phi$$

for all $A \in \mathfrak{q}$, where on the left hand side, $\tilde{A}$ denotes the fundamental vector field on the trivial Q-PFB while on the right hand side, $\tilde{A}$ is the vector field on $\mathcal{P}^H$ induced by the Cartan connection $\tilde{\omega}$. The technical details can be read in full in Čap's proof, and all carry over to the situation given by the assumptions of our proposition. Note the (potentially confusing) notational discrepancy, that our P^H corresponds to the group Q in [13], while our group Q corresponds to P there. Otherwise, only one detail of that proof must be changed in our situation, to take into account that our subgroup P^H is not necessarily parabolic in H. Namely, the open neighborhood V_1 of zero in $\mathfrak{p} \cap \mathfrak{q}_-$ there must be taken (with our, different notation), to be an open neighborhood of zero in $\mathfrak{q} \cap \mathfrak{g}_-$.

Finally, in the proof of Theorem 2.7 of [13], the local diffeomorphisms Φ above are used, under the assumption that $\kappa^{\tilde{\omega}}(u)(X, .) = 0$ for all $X \in \mathfrak{q}/\mathfrak{p}^H$, to induce a connection ω' on the trivial Q-PFB of the local leaf space, such that $\Phi^*\tilde{\omega} = \omega'$, extending $\Phi^*\tilde{\omega}$ to a one-form on the full Q-PFB via equivariance. This part of the proof, establishing the local isomorphism needed in our proposition, relies in no way on P^H being parabolic in H, and we refer the reader to it for verification of the properties of a Cartan connection. □

3.4 On the possible irreducible transitive conformal holonomy groups

This section is supplementary and not integral to the other results in our work. Here, we indicate how one can reasonably hope to get a finite classification of the fundamental (i.e. "irreducible") conformal geometries. Obviously, an important ingredient of any such classification is a finite list of possible holonomy groups. An outline of how to determine all connected, irreducible and ***transitive*** subgroups of $SO(p+1, q+1)$ is presented here, making use of results from the theory of transitive transformation groups, cf. [53].

Our strategy for generating this list is to reduce the problem to the classification results due to [38] (cf. Section 18.6 of [53]) and [32] describing the structure of Lie groups acting transitively on the product of two spheres. The reason these results are relevant is the following:

Proposition 58 *For $p, q \geq 1$, there is a two-fold covering of the Möbius sphere by the product of two spheres:*

$$S^p \times S^q \to S^{p,q}.$$

For $p, q \geq 2$, this is the universal covering of the Möbius sphere.

Proof: The Möbius sphere $S^{p,q}$ is by definition the set of all real lines in $\mathbb{R}^{p+q+2}$ which are isotropic (null) with respect to $(,)_{p+1,q+1}$, here we calculate with the *standard* metric of signature $(p+1, q+1)$. If we denote by $[x]$ the equivalence class of a vector $x \in \mathbb{R}^{p+q+2}$ under the action of $\mathbb{R}^*$ giving the real projectivization, then

$$S^{p,q} = \{\, [x] \,|\, \mathbb{R}^{p+q+2} \ni x \neq 0, (x, x)_{p+1,q+1} = 0\}.$$

Now consider the space

$$\tilde{S^{p,q}} := \{\, [[x]] \,|\, \mathbb{R}^{p+q+2} \ni x \neq 0, (x, x)_{p+1,q+1} = 0\},$$

where $[[.]]$ denotes the equivalence class under the action of $\mathbb{R}^+$, i.e. $x \sim y$ iff there exists $\lambda \in \mathbb{R}^+$ s.t. $x = \lambda y$. Then $\tilde{S^{p,q}}$ is clearly a two-fold covering of the Möbius sphere, and we claim $\tilde{S^{p,q}} \approx S^p \times S^q$.

Let $[[x]] = \mathbb{R}^+ \cdot x \in \tilde{S^{p,q}}$. Then write

$$x = \hat{x} + \hat{\hat{x}} \in \mathbb{R}^{p+1} \oplus \mathbb{R}^{q+1},$$

with $(\hat{x}, \hat{x})_{p+1} = (\hat{\hat{x}}, \hat{\hat{x}})_{q+1}$. Then the following map is well-defined:

$$\tilde{S^{p,q}} \to \mathbb{R}^{p+q+2}$$

$$\mathbb{R}^+ \cdot x \mapsto \sqrt{\frac{2}{(x, x)_{p+q+2}}} \cdot x;$$

and from

$$\sqrt{\frac{2}{(x, x)_{p+q+2}}} = \sqrt{\frac{2}{(\hat{x}, \hat{x})_{p+1} + (\hat{\hat{x}}, \hat{\hat{x}})_{q+1}}},$$

we have

$$\mathbb{R}^+ \cdot x \mapsto \left(\frac{\hat{x}}{|\hat{x}|_{p+1}}, \frac{\hat{\hat{x}}}{|\hat{\hat{x}}|_{q+1}}\right) \in S^p \times S^q,$$

giving the required diffeomorphism. $\square$

Let $H \subseteq SO(p+1, q+1)$ act transitively on $S^{p,q}$. Then a standard result, cf. Proposition 6 in Chapter 1 of [53] says that there exists a transitive action by $\tilde{H}$, the universal covering of H, on $S^p \times S^q$, which covers the action of H on $S^{p,q}$. First let us consider transitive subgroups of $SO(p+1, q+1)$ for $p, q \geq 2$, which turns out to be simpler. B. Kamerich proved in his thesis [38] a classification of the transitive and *irreducible* actions of compact Lie groups on the product of two spheres of this type. Beware that "irreducible" in this context has a distinct meaning:

Definition 59 *Let X be the homogeneous space of a compact, connected Lie group K. The action of K on X is* irreducible *if K has no proper normal transitive subgroup.*

To emphasize this distinction, we'll always say that a subgroup $H \subseteq SO(p+1, q+1)$ *acts irreducibly* when we mean Definition 59, and we'll say that such a subgroup H is *irreducible with respect to the standard representation* to mean that H leaves no proper subspace of $\mathbb{R}^{p+q+2}$ invariant. Kamerich's result, as presented in [53] (Theorem 6 in Chapter 5), is as follows:

Theorem 60 *(Kamerich [38], 1977)*
If G is a connected, compact Lie group acting transitively and irreducibly on $S^p \times S^q \approx G/H$, for $p, q \geq 2$, then one of the following holds:

(i) G is simple and the action of G is locally similar to one of the following actions:

$$\begin{aligned} SU(4)/SU(2) &\cong S^5 \times S^7; \\ Spin(8)/G_2 &\cong S^7 \times S^7 \\ Spin(7)/SU(3) &\cong S^6 \times S^7; \\ &\cong Spin(8)/SU(4) \\ &\cong SO(8)/SO(6). \end{aligned}$$

(ii) G is not simple, and the action of $G = G_1 \times G_2$ on $S^p \times S^q$ is split, i.e. G_1 and G_2 act transitively on S^p and S^q, respectively. Then the action is locally similar to the product of any two of the following actions:

$$\begin{aligned} SO(n)/SO(n-1) &\cong S^{n-1}; \\ SU(n)/SU(n-1) &\cong S^{2n-1}; \\ Sp(n)/Sp(n-1) &\cong S^{4n-1}; \\ Spin(9)/Spin(7) &\cong S^{15}; \\ Spin(7)/G_2 &\cong S^7; \\ G_2/SU(3) &\cong S^6. \end{aligned}$$

(iii) The action of G is locally isomorphic to one of the pairs, where the subgroups H_{12} are described in Table 12, Chapter 5 of [53]:

$$\begin{aligned} (SU(2) \times SU(2))/H_{12} &\cong S^3 \times S^2; \\ (SU(2n+1) \times SU(2))/SU(2n) \cdot H_{12} &\cong S^{4n+1} \times S^2; \\ (Sp(n) \times SU(2))/Sp(n-1) \cdot H_{12} &\cong S^{4n-1} \times S^2; \\ (Sp(n) \times SU(3))/Sp(n-1) \cdot Sp(1) &\cong S^{4n-1} \times S^5; \\ (Sp(n) \times Sp(2))/Sp(n-1) \cdot Sp(1) &\cong S^{4n-1} \times S^7. \end{aligned}$$

All of these give rise to an irreducible transitive action on a product of two spheres.

There are two obstacles to applying this result to our situation. First, this gives a list of all ***compact*** and connected Lie groups acting transitively on $S^p \times S^q$, while the transitive subgroups of $SO(p+1, q+1)$ which are irreducible with respect to the standard representation, are in general non-compact. However, for $p, q \geq 2$ the product $S^p \times S^q$ is simply connected and we may apply:

Proposition 61 *([48], 1950)*
Let a Lie group G act transitively on a compact, simply connected manifold. Then all maximal compact subgroups $K \subseteq G$ also act transitively on the manifold.

Thus we know that for any transitive subgroup $H \subseteq SO(p+1, q+1)$, the maximal compact subgroup of $\tilde{H}$ must appear in the list of Theorem 60.

Secondly, we have to take into account that this result only lists the irreducibly acting transitive groups. Let K be a Lie group acting transitively on a homogeneous space X. It is a fact that there always exists a normal connected, transitively acting subgroup $G \subseteq K$ such that the action of G on X is irreducible. Moreover, classifying the irreducibly acting transitive groups leads to a classification of all transitively acting groups as follows (cf. Proposition 3.6 of [41]): Let $G \subset K$ be a proper normal subgroup which acts transitively and irreducibly, and let $L \subset K$ be a normal complement, i.e. $K = G \cdot L$. Then

$$L \subseteq \mathrm{Cen}_{\mathrm{Sym}(X)}(G) = C$$

is contained in the group C of all permutations of X which centralize G. If we let G_x be the isotropy subgroup of a point $x \in X$ (so that $X \approx G/G_x$), and let $N = \mathrm{Nor}_G(G_x)$ be the normalizer of the isotropy subgroup in G, then there is an isomorphism

$$N/G_x \cong C. \tag{3.15}$$

Hence it suffices to know all the irreducibly acting, transitive groups G and determine the G-normalizer of the corresponding isotropy group.

The following Lemma will be useful for generating our list of possible irreducible conformal holonomy groups which acts transitively, and it is also interesting in its own right:

Lemma 62 *Let $Hol(M, c)$ be a connected conformal holonomy group which is irreducible with respect to the standard representation. Then $Hol(M, c)$ is semisimple.*

Proof: We may work at the level of Lie algebras, since $Hol(M, c)$ is assumed to be connected. A classical result of Cartan, cf. [52], states that irreducible real representations of real Lie algebras are of one of two types:

Proposition 63 *(Cartan, [27]) Let $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$ be an irreducible representation of a real Lie algebra $\mathfrak{g}$ on a real n-dimensional vector space V. Then either*
(I) The complexification

$$\rho^{\mathbb{C}} : \mathfrak{g} \to V \otimes \mathbb{C}$$

is irreducible, in which case the complexification $\mathfrak{g}(\mathbb{C})$ is an irreducible complex Lie subalgebra of $\mathfrak{sl}(V \otimes \mathbb{C}) \cong \mathfrak{sl}(n, \mathbb{C})$; or
(II) The vector space V is the underlying real vector space of a complex vector space V' of complex dimension $\frac{n}{2}$, and ρ is the underlying real representation of an irreducible complex representation

$$\rho' : \mathfrak{g} \to \mathfrak{gl}(V').$$

These two types are mutually exclusive.

Applying this result to our situation, we see that if the irreducible representation given by $\mathfrak{hol}(M, c) \subseteq \mathfrak{so}(p + 1, q + 1)$ is not of type (I), then in particular

$$\mathfrak{hol}(M, c) \subseteq \mathfrak{gl}(\frac{p + q + 2}{2}, \mathbb{C}) \cap \mathfrak{so}(p + 1, q + 1) = \mathfrak{u}(\frac{p + 1}{2}, \frac{q + 1}{2}).$$

But by the main result of [46], this implies that $\mathfrak{hol}(M, c) \subseteq \mathfrak{su}(\frac{p+1}{2}, \frac{q+1}{2})$. In particular, $\mathfrak{hol}(M, c)$ is an irreducible real Lie subalgebra of $\mathfrak{sl}(\frac{p+q+2}{2}, \mathbb{C})$. This means it is either an irreducible complex Lie subalgebra of $\mathfrak{sl}(\frac{p+q+2}{2}, \mathbb{C})$ or the underlying real algebra of such. In either case, the irreducible complex Lie subalgebras of $\mathfrak{sl}(m, \mathbb{C})$ are all known, also by a result of Cartan in [26], to be semi-simple, and thus $\mathfrak{hol}(M, c)$ is also semi-simple. This last argument also shows that all irreducible conformal holonomy algebras $\mathfrak{hol}(M, c)$ of type (I) must be semi-simple. $\square$

Finally, we can use the above results to begin enumerating the transitive, irreducible conformal holonomy groups which could appear. Let $Hol(M,c) \subseteq SO(p+1,q+1)$ be a connected transitive conformal holonomy group which is irreducible with respect to the standard representation. Taking $p \leq q$ and $p+q > 3$, suppose that either $p \neq 1$ or $q = 2m$. The results cited above allow us to give a procedure for enumerating all possible transitively acting conformal holonomy groups in these signatures which are irreducible with respect to the standard representation.

The procedure for getting this list is as follows. Suppose $Hol(M,c)$ meets the assumptions given. Then the universal covering group $\tilde{H}$ of $Hol(M,c)$ acts transitively on $S^p \times S^q$. If $p = 1$ and $q = 2m$, then this covering must be one of the groups in Theorem 2 of [32], but none of these has a Lie algebra which is irreducible as a subalgebra of $\mathfrak{so}(2, 2m+1)$ with respect to the standard representation.

If $p \geq 2$, then by Proposition 61, the maximal compact subgroup of $\tilde{H}$ also acts transitively on $S^p \times S^q$, and therefore must be locally equivalent to one of the transitive groups listed in Theorem 60, or else (if the maximal compact subgroup doesn't act irreducibly on $S^p \times S^q$) to a compact group containing one of these as a normal subgroup. Looking at the Lie algebras of these compact groups, we check to see which can occur as the maximal compact subalgebra of a conformal holonomy algebra $\mathfrak{hol}(M,c) \subseteq \mathfrak{so}(p+1,q+1)$ which is irreducible with respect to the standard representation on $\mathbb{R}^{p+q+2}$.

Note that it is possible to check this in a finite number of steps, since $\mathfrak{hol}(M,c)$ is semisimple by Lemma 62, and the maximal compact subalgebra of a semisimple Lie algebra is unique up to conjugation and are known by the classification of simple real Lie algebras. In particular, since all Lie algebras corresponding to the groups listed in Theorem 60 are either simple or the product of two simple groups, we note that the same must hold for $\mathfrak{hol}(M,c)$.

Thus the complete list of possibilities, for $p, q \geq 2$, can be obtained by going through the list of all real simple Lie algebras and all semisimple algebras of length 2 (i.e. direct sums of two simple Lie algebras), and seeing which have a maximal compact subalgebra corresponding to one of the listed, transitively acting groups. For those algebras $\mathfrak{h}$ which do, we have to see if there exists an irreducible representation of $\mathfrak{h}$ on $\mathbb{R}^{p+1,q+1}$ which embeds $\mathfrak{h}$ as a subalgebra of $\mathfrak{so}(p+1,q+1)$. Using the methods in [52] and tables of information on simple Lie algebras and the irreducible representations of their fundamental weights such as those in [54], this can be done with a bit of work.

The following groups all occur, but more work needs to be done to establish which other groups arising via Theorem 60 can be irreducibly embedded in the appropriate $SO(p+1,q+1)$, and to exclude the others - i.e., to complete the list:

$SO_0(p+1, q+1)$;

$SU(\frac{p+1}{2}, \frac{q+1}{2})$;

$Sp(\frac{p+1}{4}, \frac{q+1}{4})$;

$Sp(1) \cdot Sp(\frac{p+1}{4}, \frac{q+1}{4})$;

$\mathbb{T} \cdot Sp(\frac{p+1}{4}, \frac{q+1}{4})$;

$SO(p+1, \mathbb{C}) \subset SO_0(p+1, p+1)$;

$SO(p+1, \mathbb{H}) \subset SO_0(p+1, p+1)$;

$Spin(9, \mathbb{C}) \subset SO_0(16, 16)$;

$Spin(1, 8) \subset SO_0(8, 8)$;

$Spin(7, \mathbb{C}) \subset SO_0(8, 8)$;

$G_2(\mathbb{C}) \subset SO_0(8, 8)$;

$Spin(3, 4) \subset SO_0(4, 4)$;

$G_{2,2} \subset SO_0(3, 4)$;

$Sp(n) \times SU(2) \subset SO_0(4n, 6)$;

$Sp(n) \times Sp(2) \subset SO_0(4n, 8)$;

$SU(2) \times SU(2) \subset SO_0(4, 5)$.

Chapter 4

Fefferman spaces with special conformal holonomy

We are now prepared to establish the conformal holonomy correspondence for the irreducible group $Sp(p''+1,q''+1) \subset SO_0(p+1,q+1)$, where we adopt for this Chapter the notation conventions $p'' := \frac{p-3}{4}, q'' := \frac{q-3}{4}, n'' = p''+q''$ and also, $p' := \frac{p-1}{2}, q' := \frac{q-1}{2}, n' = p'+q'$.

By conformal holonomy correspondence, in general for any of the groups H listed in Chapter 3.4, this means that for some parabolic subgroup $Q \subset H$ and a certain class of parabolic geometries of type (H,Q) we have the following: On the one hand, the normal, regular parabolic geometries of type (H,Q) from this class induce conformal Fefferman spaces with special holonomy contained in H. On the other hand, any conformal manifold with special holonomy contained in H, is locally isomorphic to a conformal Fefferman space of the canonical parabolic geometry of such a structure.

This result will be proved in Chapter 4.2. Our method of proof is based on an adaptation/generalization of that used to establish the analogous correspondence for $SU(p'+1,q'+1)$ in [18] and [19].

4.1 CR and QC geometry

4.1.1 The standard definitions of CR and QC structures

First, we want to introduce the parabolic geometries which will appear as the base spaces of the conformal Fefferman space having these holonomy groups: For $SU(p'+1,q'+1)$, these are torsion-free (or equivalently, integrable) CR structures; for $Sp(p''+1,q''+1)$, we get quaternionic contact (QC) structures. These two parabolic geometries can be thought of as the complex and quaternionic analogs of conformal geometry (cf. Introduction, [6]), a viewpoint which is strengthened by the conformal holonomy correspondences in both cases. To begin this Section, we recall the standard notion of a CR structure, and Biquard's definition of QC structures, and some of the geometric objects which CR and QC manifolds are equipped with. We then describe the homogeneous model spaces and parabolic structures associated to both.

Definition 64 *Let N be a smooth manifold of odd dimension $2n'+1$. An* almost CR structure *on N is given by a co-dimension one distribution $\mathcal{V} \subset TN$ and an almost complex structure J on $\mathcal{V}$, i.e. a tensor $J \in \mathcal{V}^* \otimes \mathcal{V}$ such that $J^2 = -\mathrm{Id}$. An* almost CR manifold *is given by such a triple $(N, \mathcal{V}, J)$. An* (integrable) CR manifold *is an almost CR manifold $(N, \mathcal{V}, J)$ such that the the Nijenhuis tensor N_J is well-defined and vanishes: For all $X, Y \in \Gamma(\mathcal{V})$, we have $[JX, Y] + [X, JY] \in \Gamma(\mathcal{V})$, and*

$$N_J(X,Y) := J([JX,Y] + [X,JY]) - [JX,JY] + [X,Y] \equiv 0.$$

Alternatively, an almost CR structure on N may be defined as a complex subbundle $T^{1,0}$ of $TN^{\mathbb{C}} = TN \otimes \mathbb{C}$ of complex dimension n', such that

$$T^{1,0} \cap T^{0,1} = \{0\},$$

where $T^{0,1} := \overline{T^{1,0}}$. It is a standard fact this is equivalent to Definition 64, under the following correspondence: In one direction, take $\mathcal{V} := Re(T^{1,0} \oplus T^{0,1})$ and $J(U + \bar{U}) := i(U - \bar{U})$; in the other direction, take $T^{1,0}$ to be the eigenspace of i for complex-linear extension $J^{\mathbb{C}} : \mathcal{V} \otimes \mathbb{C} \to \mathcal{V} \otimes \mathbb{C}$. The integrability condition $N_J \equiv 0$ is expressed in terms of the second definition as integrability of the subbundle $T^{1,0}$:

$$[\Gamma(T^{1,0}), \Gamma(T^{1,0})] \subseteq \Gamma(T^{1,0}).$$

We will make use of this correspondence, as the different definitions are useful for different purposes. In particular, the description of an almost CR structure via the complex subbundle $T^{1,0}$ is useful as it gives a decomposition of alternating forms on M, by defining

$$\begin{aligned}
\Lambda^{1,0} &:= (T^{0,1})^{\perp} \subset T^*N \otimes \mathbb{C};\\
\Lambda^{0,1} &:= (T^{1,0})^{\perp};\\
\Lambda^{r,s} &:= \Lambda^r(\Lambda^{1,0}) \otimes \Lambda^s(\Lambda^{0,1}).
\end{aligned}$$

In particular, we have the complex line bundle $\mathcal{K} := \Lambda^{n'+1,0}$, called the *canonical bundle* of $(N, \mathcal{V}, J)$.

Definition 65 *Given an almost CR manifold $(N, \mathcal{V}, J)$, its* Levi form *is defined by:*

$$\begin{aligned}
L &: \mathcal{V} \times \mathcal{V} \to (TN/\mathcal{V})\\
L &: (X,Y) \mapsto [JX, Y] + \mathcal{V}
\end{aligned}$$

An almost CR manifold is non-degenerate *if its Levi form L is (point-wise) non-degenerate.*
A pseudo-Hermitian form *for $(N, \mathcal{V}, J)$ is a non-vanishing one-form $\theta \in \Omega^1(N)$ satisfying $\theta_{|\mathcal{V}} \equiv 0$.*

We will assume that globally defined pseudo-Hermitian forms exist for the almost CR manifolds we consider, which is equivalent to orientability of the manifold. Given a pseudo-Hermitian form θ, its Levi form L_θ is the real-valued symmetric tensor given by $\theta \circ L : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, and L_θ satisfies

$$L_\theta(X,Y) = d\theta(X, JY). \tag{4.1}$$

If $(N, \mathcal{V}, J)$ is non-degenerate, then a pseudo-Hermitian form for it is necessarily a contact form, i.e. $\theta \wedge (d\theta^{n'})$ is non-vanishing. In particular, there is a uniquely defined vector field $\xi \in \Gamma(TN)$ – the *Reeb vector field* for θ – which is complementary to $\mathcal{V}$, defined by the conditions

$$\theta(\xi) \equiv 1; \quad \xi \lrcorner d\theta \equiv 0.$$

The Levi-form L_θ for a pseudo-Hermitian form of a non-degenerate almost CR manifold is non-degenerate, and has a signature (p', q') for $p' + q' = n'$. Moreover, we fix a positive orientation and consider only positively-oriented pseudo-Hermitian forms, so this signature is well-defined. We call such a structure $(N, \mathcal{V}, J)$ a *non-degenerate almost CR manifold of signature* (p', q'). Typical examples of integrable CR structures are given by generic real hypersurfaces of complex manifolds. Any Sasakian manifold (M, η, g) (cf. the book [8]) induces a CR structure by varying the contact form η by a conformal factor.

QC structures are the quaternionic analog of CR structures, cf. Definition 2.1 of [37]:

Definition 66 *(Biquard [6]) A quaternionic contact (QC) structure of signature* (p'', q'') *on a* $4n'' + 3$ *dimensional manifold* N', $n'' = p'' + q'' \geq 1$, *is the data of a codimension three distribution* $\mathcal{V}'$ *on* N' *equipped with a* $CSp(1)Sp(p'', q'')$ *structure, i.e. we have:*
i) a fixed conformal class $[g]$ *of metrics on* $\mathcal{V}'$ *of signature* $(4p'', 4q'')$*;*
ii) a 2-sphere bundle $\mathbb{Q}$ *over* N' *of almost complex structures, such that, locally we have* $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 | a^2 + b^2 + c^2 = 1\}$, *where the almost complex structures* $I_s : \mathcal{V}' \to \mathcal{V}'$, $I_s^2 = -Id$, $s = 1, 2, 3$, *satisfy the commutation relations of the imaginary quaternions* $I_1 I_2 = -I_2 I_1 = I_3$*;*
iii) $\mathcal{V}'$ *is locally the kernel of a one-form* $\eta = (\eta_1, \eta_2, \eta_3)$ *with values in* $\mathbb{R}^3$ *and the following compatibility condition holds:*

$$g(X, Y) = d\eta_i(X, I_i Y), \ s = 1, 2, 3, \ X, Y \in \mathcal{V}'. \tag{4.2}$$

In dimension seven ($n'' = 1$)*, a QC structure is additionally assumed to satisfy the following* integrability condition *(cf. Definition 1.3 of [30]): There exists a local oriented orthonormal basis* $(d\eta_i|_{\mathcal{V}'})$ *of* $\Lambda^2_+(\mathcal{V}')^*$ *and vector fields* ξ_1, ξ_2, ξ_3 *satisfying*

$$\xi_i \lrcorner \eta_j = \delta_{ij}; \ \ \xi_i \lrcorner d\eta_j|_{\mathcal{V}'} = -\xi_j \lrcorner d\eta_i|_{\mathcal{V}_i'}. \tag{4.3}$$

Note that the extra integrability in dimension seven is required to define QC structures, since a $CSp(1)Sp(1)$ structure is just a conformal metric structure on $\mathcal{V}'$. In higher dimensions, the existence (and uniqueness!) of local vector fields ξ_i satisfying (4.3) is automatic. We refer to these ξ_i as the *Reeb vector fields* for the QC contact form η, in analogy to the CR case.

The canonical example of a QC structure is the sphere S^{4n-3}, $n > 1$, considered as the quaternionic conformal infinity of quaternionic hyperbolic space of dimension n, which is locally equivalent to the quaternionic Heisenberg group as a QC manifold. Any totally umbilical, real hypersurface of a quaternionic Kähler or hyper-Kähler manifold, naturally inherits a QC structure. Also, any locally 3-Sasakian manifold can be considered as a QC manifold, by allowing the contact structure to vary conformally in the obvious manner indicated by Definition 66.

4.1.2 CR and QC structures as parabolic geometries

Some more of the classical geometric objects for CR structures will be discussed in Chapter 4.4 in relating the conformal Fefferman space of Chapter 3.3 to the classical Fefferman construction for CR manifolds. For more on the corresponding constructions in QC geometry, such as the Biquard connection, cf. either [6] or [37]. In this Section, we wish to describe the realization of both of these geometries as parabolic geometries.

To see this, and also to get a simple picture of the basic geometry behind the Fefferman construction in these cases, we start with the homogeneous model of these parabolic geometries. For strictly pseudo-convex CR manifolds of dimension $2n'+1$, the homogeneous model is $S^{2n'+1}$, which can be viewed as the conformal infinity of complex hyperbolic space $\mathbb{C}\mathbf{H}^{n'+1}$ (cf. Definition B of [6]). For CR structures of arbitrary signature (p', q'), consider the semi-Hermitian vector space

$$(\mathbb{C}^{p'+1,q'+1}, <,>^{\mathbb{C}}_{p'+1,q'+1})$$

of complex dimension $p'+q'+2$, with the Hermitian metric $<,>^{\mathbb{C}}_{p'+1,q'+1}$ (abbreviated by $<,>^{\mathbb{C}}$) induced by the form $I^{1,1}_{p',q'}$, cf. (3.1) in Chapter 3.1. Then, analogous to the construction in that Chapter, we take $C^{p',q'}_{\mathbb{C}}$ to be the light cone with respect to this metric, and take the homogeneous model space to be its complex projectivization:

$$p_{\mathbb{C}} : C^{p',q'}_{\mathbb{C}} \to \mathbb{CP}(C^{p',q'}_{\mathbb{C}}) =: S^{p',q'}_{\mathbb{C}}.$$

A CR structure on this space is given naturally: For $x = [v] \in S^{p',q'}_{\mathbb{C}}$, for some $v \in C^{p',q'}_{\mathbb{C}}$, take

$$\mathcal{V}_x := (p_{\mathbb{C}})_*(v^{\perp_{\mathbb{C}}}) \subset T_x S^{p',q'}_{\mathbb{C}}.$$

Then $\mathcal{V}$ defines in this way a co-dimension 1 contact distribution on $S^{p',q'}_{\mathbb{C}}$ and it can be seen that complex multiplication on $\mathbb{C}^{p'+1,q'+1}$ induces an integrable complex structure J on this distribution, which has a non-degenerate Levi form of signature (p', q'). To see this, choose a section $\sigma : S^{p',q'}_{\mathbb{C}} \to C^{p',q'}_{\mathbb{C}}$ of the projection given by complex projectivization; pulling back the restriction of the hermitian metric $<,>^{\mathbb{C}}$ to the complex orthogonal subspace defining $\mathcal{V}$, gives a non-degenerate metric on $\mathcal{V}$ of signature (p', q'). Thus

$$(S^{p',q'}_{\mathbb{C}}, \mathcal{V}, J)$$

defines a pseudo-convex CR manifold of signature (p', q'). In particular, note that in the strictly pseudo-convex case (i.e. when the Levi form is positive definite), we have $S^{n'}_{\mathbb{C}} = S^{2n'+1}$.

If we let $H := SU(p'+1, q'+1)$ and let $Q := \mathrm{stab}_H(L_{\mathbb{C}})$, where $L_{\mathbb{C}} := \mathbb{C}e_0$ is the complex span of the first standard basis vector of $\mathbb{C}^{p'+q'+2}$ (which is null with respect to $<,>^{\mathbb{C}}$), then

$$S^{p',q'}_{\mathbb{C}} = H/Q. \tag{4.4}$$

However, the automorphism group of the CR manifold $(S^{p',q'}_{\mathbb{C}}, \mathcal{V}, J)$ is only $PSU(p'+1, q'+1) \cong H/Z(H)$, which has H as a $(p'+q'+2)$-fold (universal) covering since $Z(H) \cong \mathbb{Z}_{p'+q'+2}$. Thus, to fix H as the automorphism group of the homogeneous geometry, we need to fix in addition to the CR structure a discrete geometric structure, and we'll refer to the resulting geometric structure as a CR^+ structure. In the homogeneous case, this is clearly given by including the entire (tautological) complex line bundle over $S_{\mathbb{C}}^{p',q'}$ given via the complex projectivization map. This bundle is a $(p'+q'+2)$nd root of the canonical complex line bundle determined by the CR structure.

Using this homogeneous model of CR geometry, we also get a nice geometric picture of how the conformal Fefferman space arises from it. Namely, if we consider $\mathbb{C}^{p'+q'+2}$ as a real vector space, and let $<,>^{\mathbb{R}} = Re <,>^{\mathbb{C}}$, we get pseudo-Euclidean space of signature $(2p'+2, 2q'+2) = (p+1, q+1)$, and the light cone with respect to this real background, $C^{p+1,q+1}$, is the same as $C_{\mathbb{C}}^{p'+1,q'+1}$. If we take the real instead of the complex projectivization, this gives us the homogeneous model for conformal geometry, $S^{p,q} = S^{2p'+1,2q'+1}$, giving a fibration over $S_{\mathbb{C}}^{p',q'}$ with fiber $S^1 \cong Q/P^H$.

From the description of the pair (H, Q) above, we see that the Lie algebra $\mathfrak{h}$ of H is given a $|2|$-grading by defining:

$$\mathfrak{h}_{-2} := \{\mathfrak{h}_{-2}(ia) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ia & 0 & 0 \end{pmatrix} \mid a \in \mathbb{R}\};$$

$$\mathfrak{h}_{-1} := \{\mathfrak{h}_{-1}(X) := \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -\bar{X}^{\psi*} & 0 \end{pmatrix} \mid X \in \mathbb{C}^{p',q'}\};$$

$$\mathfrak{h}_0 := \{\mathfrak{h}_0(a, A) := \begin{pmatrix} a+ib & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a+ib \end{pmatrix} \mid a \in \mathbb{R}, A \in \mathfrak{u}(p',q'), b = -\frac{1}{2}\mathrm{tr}_{\mathbb{C}}A\};$$

and $\mathfrak{h}_1 = (\mathfrak{h}_{-1})^*; \mathfrak{h}_2 = (\mathfrak{h}_{-2})^*$. In particular, the maps $\mathfrak{h}_i$ for $-2 \leq i \leq 2$ defined in this way, give an isomorphism of the $|2|$-grading:

$$\begin{aligned} \mathfrak{h} &= \mathfrak{h}_{-2} \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \\ &\cong Im(\mathbb{C}) \oplus \mathbb{C}^{p',q'} \oplus (\mathbb{R} \oplus \mathfrak{u}(p',q')) \oplus (\mathbb{C}^{p',q'})^* \oplus (Im(\mathbb{C}))^*. \end{aligned}$$

From this we see that a regular parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of type (H, Q) gives a filtration of the tangent bundle

$$TN = T^{-2}N \supset T^{-1}N \supset \{0\}$$

with $T^{-1}N$ an even dimensional distribution of co-dimension one, a pointwise isomorphism of $\mathrm{gr}(TN)$ to the complex Heisenberg algebra $\mathfrak{h}_- = Im(\mathbb{C}) \oplus \mathbb{C}^{p',q'}$, and a reduction of the adapted frame bundle to a structure group Q_0 with Lie algebra $\mathfrak{h}_0 \cong \mathbb{R} \oplus \mathfrak{u}(p',q')$. In particular, this gives an almost CR structure on N with by letting $\mathcal{V} = T^{-1}N$, and taking J to be the almost complex structure on $\mathcal{V}$ induced by the pointwise isomorphism $\mathcal{V} \cong \mathbb{C}^{p',q'}$.

More precisely, as is shown in 4.14-15 of [21], regular infinitesimal flag structures of type $(\mathfrak{h}, \mathfrak{q})$ correspond precisely to partially integrable almost CR structures:

Definition 67 *An almost CR structure* $(N, \mathcal{V}, J)$ *is* partially integrable *if, for all* $X, Y \in \Gamma(\mathcal{V})$*, we also have*

$$[X, Y] - [JX, JY] \in \Gamma(\mathcal{V}).$$

It is a matter of unwinding the definitions to show that the Levi form of Definition 65 for an almost CR manifold $(N, \mathcal{V}, J)$ corresponds via (4.1) to the generalized Levi form given by (2.46) in Chapter 2.3 on the associated graded tangent bundle gr(TN) to the filtration of TN by $\mathcal{V}$. And the Levi form thus defined corresponds to the algebraic bracket of the complex Heisenberg algebra if and only if the almost CR structure is partially integrable.

For partially integrable CR structures, canonical (parabolic) Cartan geometries thus always exist, and 4.16 of [21] shows that the canonical Cartan connection is torsion-free if and only if the associated CR structure is integrable. We will assume integrability of CR structures from here on. But as is evident from the homogeneous model, one needs more structure to guarantee that a parabolic geometry of type (H, Q) exists, and in general only a geometry of type $(PSU(p'+1, q'+1), PQ)$ is guaranteed.

To deal with this problem, we need the existence of a $U(1)$-PFB on which $Z(H)$ acts effectively, which is given by fixing a $(p'+q'+2)$nd root of the canonical bundle $\mathcal{K}$ of the CR manifold. Such a bundle always exists if we either work locally, or restrict, e.g. to CR manifolds given as real hypersurfaces of $\mathbb{C}^{n'+1}$. Unfortunately, the topological obstructions to the global existence of such a line bundle are not clear in general. For $p'+q'$ even, for example, this implies the existence of a spin structure. In any case, we will assume the existence of such a structure, which is needed for the conformal Fefferman construction, and call the corresponding geometry a (integrable) CR^+ structure.

The description of quaternionic contact (QC) structures as parabolic structures is quite analogous, and we will therefore be brief. For the homogeneous model, simply repeat the above steps, but starting with the quaternionic arithmetic vector space $\mathbb{H}^{p''+q''+2}$ equipped with the standard skew-hermitian metric $<,>^{\mathbb{H}}$ of signature $(p''+1, q''+1)$. Then taking $H' := Sp(p''+1, q''+1)$ and $Q' := \mathrm{stab}_{H'}(L_{\mathbb{H}})$ to be the stabilizer of the (null) quaternionic line $L_{\mathbb{H}} := \mathbb{H}e_0$, the homogeneous space H'/Q' has a natural QC structure of signature (p'', q'') defined completely analogously to how the CR structure was defined on the homogeneous model above. Again we have the problem of dealing with the center $Z(H')$, which we'll deal with below, and here we have $Z(Sp(p''+1, q''+1)) \cong \mathbb{Z}_2$.

A QC structure of signature (p'', q'') can also be defined from the infinitesimal structure of this parabolic homogeneous model. From the geometric description of H' and Q', it is again straightforward to check that we get a $|2|$-grading of the Lie algebra $\mathfrak{h}'$ of H' by defining:

$$\mathfrak{h}'_{-2} = \{\mathfrak{h}'_{-2}(x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \mid x \in Im(\mathbb{H})\};$$

$$\mathfrak{h}'_{-1} = \{\mathfrak{h}'_{-1}(X) := \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -\bar{X}^{\psi t} & 0 \end{pmatrix} \mid X \in \mathbb{H}^{p',q'}\};$$

$$\mathfrak{h}'_0 = \{\mathfrak{h}'_0(a, A) := \begin{pmatrix} a+b & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a+b \end{pmatrix} \mid a \in \mathbb{R}, A \in \mathfrak{sp}(p'', q''), b \in Im(\mathbb{H})\};$$

and $\mathfrak{h}'_2 = (\mathfrak{h}'_{-2})^*; \mathfrak{h}'_1 = (\mathfrak{h}'_{-1})^*$. And the maps $\mathfrak{h}'_i$ define an isomorphism

$$\begin{aligned} \mathfrak{h}' &= \mathfrak{h}'_{-2} \oplus \mathfrak{h}'_{-1} \oplus \mathfrak{h}'_0 \oplus \mathfrak{h}'_1 \oplus \mathfrak{h}'_2 \\ &\cong Im(\mathbb{H}) \oplus \mathbb{H}^{p'',q''} \oplus (\mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(p'', q'')) \oplus (\mathbb{H}^{p'',q''})^* \oplus (Im(\mathbb{H}))^*. \end{aligned}$$

Then we can define:

Definition 68 *Let N' be a smooth manifold of odd dimension $4n''+3$, $n''>1$. A* quaternionic contact (QC) structure *on N' is given by a smooth distribution $\mathcal{V}'\subset TN'$ of co-dimension 3, such that the associated graded tangent bundle* $\mathrm{gr}(TN')$ *to the filtered bundle*

$$TN' = T^{-2}N' \supset T^{-1}N' = \mathcal{V}' \supset \{0\},$$

equipped with the generalized Levi-form, is point-wise isomorphic to the quaternionic Heisenberg algebra $Im(\mathbb{H})\oplus\mathbb{H}^{p'',q''}$.

This definition is equivalent to Definition 66 given above. The reason is that a pointwise isomorphism of the graded tangent bundle as in Definition 68 already determines a regular infinitesimal flag structure of type $(\mathfrak{h}',\mathfrak{q}')$ on N', i.e. it already determines a reduction of the associated graded tangent bundle to the structure group $CSp(1)Sp(p'',q'')$. In dimension seven, the infinitesimal flag structure is not sufficient to guarantee the integrability condition (4.3).

On the other hand, a QC structure in the sense of Definition 66 determines a regular infinitesimal flag structure of type $(\mathfrak{h}',\mathfrak{q}')$. In particular, the existence of local quaternionic contact forms $\eta\in\Omega^1(M;\mathbb{R}^3)$, with $\ker(\eta)=\mathcal{V}'$ and satisfying the condition (4.2), implies that the associated graded tangent bundle $\mathrm{gr}(TN')$ with Levi bracket is pointwise isomorphic to the quaternionic Heisenberg algebra.

Since the first cohomology groups $H^1_l(\mathfrak{h}'_-,\mathfrak{h}')$ of positive homogeneity l vanish, we thus also have canonical normal and regular parabolic geometries for QC structures. Here again, we must take note of the center of $Sp(p''+1,q''+1)$, which however is only $\mathbb{Z}_2$, in order to get a canonical Cartan geometry of type (H',Q'). Geometrically, this is equivalent to requiring that the lift of the $CSp(1)Sp(p'',q'')$ structure on $\mathcal{V}'$ to a split structure group $C(Sp(1)\times Sp(p'',q''))$ (which always exists locally) exists globally. We will generally assume this, and refer to the corresponding geometric structures as QC^+ structures.

Finally, we note that the canonical (i.e. regular and normal) Cartan connection of a QC structure is always torsion-free, cf. Section 4.6 of [15]. In dimension $4n''+3>7$, this can be seen very easily. Namely, using Kostant's version of BBW, one sees that $H^2(\mathfrak{h}'_-,\mathfrak{h}')$ has exactly two irreducible components, one of homogeneity 0 and one of homogeneity 2. Hence for a regular normal Cartan connection ω'' of type (H',Q'), and κ its curvature function, we have $\kappa_H=\kappa^{(2)}$. Kostant's version of BBW also gives the generators of the irreducible components, and for $H^2(\mathfrak{h}'_-,\mathfrak{h}')$ the generator is a map from $\mathfrak{h}'_{-1}\wedge\mathfrak{h}'_{-1}$ to $\mathfrak{h}'_0$. In particular, this shows that $\kappa_H\in\Lambda^2(\mathfrak{h}'_-)^*\otimes\mathfrak{q}'$, which implies by Proposition 42 that the same holds for κ, i.e. ω'' is torsion-free.

4.2 The Fefferman construction and its converse for CR and QC structures

4.2.1 The existence results

For the rest of this and the next Section, we now fix the notation as above $G:=SO_0(p+1,q+1), P:=\mathrm{stab}_G(L_{\mathbb{R}}), H:=SU(p'+1,q'+1), Q:=\mathrm{stab}_H(L_{\mathbb{C}}), H':=Sp(p''+1,q''+1), Q':=\mathrm{stab}_{H'}(L_{\mathbb{H}})$. Then clearly, $H'\subset H\subset G$ all act transitively on $S^{p,q}$, $Q\supset P^H$ and $Q'\supset Q^{H'}\supset P^{H'}$. From the Fefferman construction described in Chapter 3.3, we therefore clearly have:

Proposition 69 *Given a CR^+ structure $(N, \mathcal{V}, J)$ of signature (p', q'), the canonical parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of type (H, Q) associated to it induces a conformal Fefferman space $(\mathcal{P}, \pi, M, \omega)$ of signature (p, q) with $Hol(\omega) \subseteq H$. Here M is a S^1 bundle over N.*

Given a QC^+ structure $(N', \mathcal{V}')$ of signature (p'', q''), its canonical parabolic geometry $(\mathcal{Q}', \pi'', N', \omega'')$ of type (H', Q') induces both a CR^+ Fefferman space $(\mathcal{Q}, \pi', N, \omega')$ of signature (p', q') and a conformal Fefferman space (also induced by the CR^+ Fefferman space) $(\mathcal{P}, \pi, M, \omega)$ of signature (p, q), with $Hol(\omega) = Hol(\omega') \subseteq H'$. M is a bundle over N' with fiber $S^3 \approx Sp(1)$.

We now want to establish that conformal holonomy groups H and H' give rise to local converse constructions of those in the first and second parts, respectively, of Proposition 69:

Proposition 70 *Given a conformal manifold (M, c) with $Hol(M, c) \subseteq H$, the canonical Cartan geometry is locally isomorphic to a conformal Fefferman space over a parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of type (H, Q), inducing a partially integrable CR^+ structure on N.*

Given a conformal manifold (M, c) with $Hol(M, c) \subseteq H'$, the canonical Cartan geometry is locally isomorphic to a conformal Fefferman construction over a parabolic geometry $(\mathcal{Q}', \pi'', N', \omega'')$ of type (H', Q'), inducing a QC^+ structure on N'.

Before proving this, we note that as a result of the normality of the induced Cartan connections in both cases, which will be shown in Chapter 4.3, it in fact holds that the locally induced structures "downstairs" are integrable (this distinction is vacuous in the QC case except in dimension seven). For the moment, though, this will not concern us.

Proof: From the Cartan reduction principle given by Proposition 55, we have Cartan reductions of the canonical Cartan geometry $(\mathcal{P}, \pi, M, \omega)$ associated to (M, c) to the groups H and H', respectively. Using the obvious notation for these Cartan reductions, by Proposition 57 of Chapter 3.3, it suffices to show, respectively, that $\kappa(u)(X, .) = 0$ for all $u \in \mathcal{P}^H$ and all $X \in (\mathfrak{q}/\mathfrak{p}^H)$, or that $\kappa(u')(X', .) = 0$ for all $u' \in \mathcal{P}^{H'}$ and all $X' \in (\mathfrak{q}'/\mathfrak{p}^{H'})$.

From the discussion in Chapters 3.1-3.2, we see that a reduction of $Hol(M, c)$ to H gives an orthogonal and parallel complex structure J on the standard conformal Tractor bundle $\mathcal{T}(M)$, while a reduction to H' gives three such complex structures, J_1, J_2, J_3 satisfying the commutation relations of the imaginary quaternions. These are in particular skew-symmetric endomorphisms of the standard Tractor bundle, and using the natural identification

$$\mathfrak{so}(\mathcal{T}(M)) = \mathcal{A}(M),$$

we may identify the complex structures given by our holonomy reduction correspond to parallel sections of the adjoint Tractor bundle $\mathcal{A}(M)$. We can therefore apply the following result, which follows from the existence of splitting operators for curved BGG sequences in the general parabolic setting (cf. Corollary 3.5 of [15], for the conformal setting, this was first proved in Proposition 2.2 of [33]):

Proposition 71 *Let $s \in \Gamma(\mathcal{A}(M))$ be a parallel section and let $\mathbf{k}$ denote the underlying vector field given by $\mathbf{k} = \Pi \circ s$. Then $\mathbf{k}$ is a conformal Killing field which also satisfies*

$$\mathbf{k} \lrcorner K^\omega = 0$$

for the curvature two-form $K^\omega \in \Omega^2(M;\mathfrak{g})$ corresponding to κ^ω.

Now consider the conformal Killing vector fields $\mathbf{j}$ and $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$ thus induced by the parallel adjoint Tractors $J \in \mathcal{A}(M)$ and $J_1, J_2, J_3 \in \mathcal{A}(M)$ given by the conformal holonomy reductions to H and H', respectively. By Proposition 71, we have:

$$\mathbf{j} \lrcorner K^\omega = 0;$$
$$\mathbf{j}_i \lrcorner K^\omega = 0;$$

respectively, for $i = 1, 2, 3$. But from the definitions of Q and Q', it is easy to see that $\mathbf{j}$ spans the distribution $\mathcal{V}^{\mathfrak{q}} = \mathcal{P}^H \times_{P^H} (\mathfrak{q}/\mathfrak{p}^H)$ induced by $\mathfrak{q} \supset \mathfrak{p}^H$, and the $\mathbf{j}_i$ span the analogous distribution $\mathcal{V}^{\mathfrak{q}'}$ induced by $\mathfrak{q}' \supset \mathfrak{p}^{H'}$. Thus the assumptions of Proposition 57 are fulfilled in both cases, proving Proposition 70. □

4.2.2 Parallel adjoint Tractors

Proposition 70 gives a quick verification that conformal holonomy reductions to the groups H and H' always guarantee the existence of a local fibration over a partially integrable CR^+ structure or over a QC^+ structure, respectively. In fact, as a result of the normality of the induced Cartan connections in both cases, which will be proved in the next section, it actually follows that the structures in both cases are integrable (for the QC case, this is only relevant in dimension 7), since the Cartan connections inducing them are automatically torsion-free. While in principle this is enough to proceed with the holonomy correspondence, in the next Section we want to give some details of how these structures are induced on the local leaf spaces, in terms of underlying geometries. In the course of that, we'll also verify a few properties which will be useful in the computations of the next Section.

First we indicate, in our situation, how the correspondence in Proposition 71 works. For this, recall that any metric $g \in c$ in our conformal class gives a P-invariant splitting

$$\mathcal{A}(M) \cong^g TM \oplus \mathfrak{co}(TM) \oplus T^*M.$$

Correspondingly, given a choice of metric g we can represent a Tractor $s \in \Gamma(\mathcal{A}(M)) = C^\infty(M, \mathfrak{g})$ in the matrix form:

$$s = \begin{pmatrix} -\alpha_s & \gamma_s & 0 \\ \mathbf{k}_s & \mathrm{K}_s & -\gamma_s^{\psi t} \\ 0 & -\mathbf{k}_s^{\psi t} & \alpha_s \end{pmatrix}$$

where $\mathbf{k}_s \in \Gamma(TM), \mathrm{K}_s \in \Gamma(\mathfrak{so}(TM)) = \Gamma(T^*M \wedge TM), \alpha_s \in C^\infty(M)$ and $\gamma_s \in \Omega^1(M)$. Evidently, the vector field $\Pi \circ s$ given by the natural projection from the adjoint bundle to the tangent bundle, is given by $\mathbf{k}_s$, and in particular this component of the representation is invariant under conformal change of the metric.

Using this decomposition, one can compute formulae for the linear connection $\nabla^{\mathcal{A}}$ and its curvature endomorphism $\mathcal{R}^{\mathcal{A}}$ with respect to this representation and decomposition, which are analogous and very similar to (3.9) and (3.10), respectively. Comparing with the formulae in Section 3 of [44], we get:

$$\nabla^{\mathcal{A}}_X s = \begin{pmatrix} \nabla^g_X & -X\lrcorner & X\wedge & 0 \\ -\mathrm{P}^g(X,.)\wedge & \nabla^g_X & 0 & X\wedge \\ \mathrm{P}^g(X,.) & 0 & \nabla^g_X & X\lrcorner \\ 0 & \mathrm{P}^g(X,.)\lrcorner & \mathrm{P}^g(X,.)\wedge & \nabla^g_X \end{pmatrix} \begin{pmatrix} \mathbf{k}_s \\ \mathrm{K}_s \\ \alpha_s \\ \gamma_s \end{pmatrix}. \tag{4.5}$$

In particular, as is computed in Section 4 of [44], the first line of the equation $\nabla^{\mathcal{A}} s = 0$ is conformally invariant, and implies:

$$\mathrm{rot}^g(\mathbf{k}_s) = 2\mathrm{K}_s;$$
$$\mathrm{div}^g(\mathbf{k}_s) = -n\alpha_s;$$

where for any vector field V, $\mathrm{rot}^g(V)$ is the skew symmetric endomorphism on TM defined by identifying $d(V^*)$ with such an endomorphism via the isomorphism given by the metric, and $\mathrm{div}^g(V) = -d^*(V^*)$. In particular, inserting these identities into the first line of (4.5) for any parallel adjoint Tractor s, gives the identity

$$\nabla^g_X \mathbf{k}_s = \frac{1}{2} X \lrcorner \mathrm{rot}^g(\mathbf{k}_s) + \frac{1}{n}\mathrm{div}^g(\mathbf{k}_s) \cdot X$$

for any vector field X and $\mathbf{k}_s = \Pi \circ s$ as above, and this is a common reformulation of the conformal Killing equation on vector fields.

To see the second part of the statement in Proposition 71, one can directly compute with the explicit, matrix form of the curvature K^ω of the canonical Cartan connection. A more conceptual approach is given via the notion of infinitesimal automorphisms, cf. [15]. Given an arbitrary parabolic geometry $(\mathcal{P}, \pi, M.\omega)$ of type (G, P) an *infinitesimal deformation* of the parabolic geometry (cf. 3.1 of [15]) is given by a family $\{\omega_\tau\}$ of Cartan connections on $\mathcal{P}$ which is smoothly parameterized by $\tau \in (-\epsilon, \epsilon) \subset \mathbb{R}$, such that $\omega_0 = \omega$. The infinitesimal deformation is an *infinitesimal automorphism* if and only if:

$$\frac{d}{d\tau}_{|\tau=0} \omega_\tau = 0.$$

A section of the adjoint Tractor bundle $s \in \Gamma(\mathcal{A}(M))$, naturally defines a vector field on $\mathcal{P}$, and the local flows of this vector field determine infinitesimal deformations of the parabolic geometry by pulling back the Cartan connection along them. The following fact (cf. Proposition 3.2 of [15]) is proven by direct application of the formulae for the (standard and adjoint) Tractor connections of a parabolic Cartan connection:

Proposition 72 *Let $s \in \mathcal{A}(M)$ and let $K \in \Omega^2(M, \mathcal{A}(M))$ be the curvature operator of the parabolic geometry $(\mathcal{P}, \pi, M, \omega)$. The infinitesimal deformation of ω induced by s is given by $\nabla^{\mathcal{A}} s + K(\Pi(s), .)$. In particular, s is an infinitesimal automorphism if and only if $\nabla^{\mathcal{A}} s = -K(\Pi(s), .)$.*

For $(\mathcal{P}, \pi, M, \omega)$ the normal Cartan geometry associated to a conformal manifold (M, c), Proposition 71 follows immediately, since infinitesimal automorphisms of the conformal structure (i.e. conformal Killing fields) are in bijective correspondence to infinitesimal automorphisms of ω, which correspond to adjoint Tractors s satisfying $\nabla^{\mathcal{A}} s = -K^\omega(\Pi(s), .)$. This defines the *splitting operator* $L : \Gamma(TM) \to \Gamma(\mathcal{A}(M))$, which is an inverse to the restriction of the natural projection operator Π to the space of infinitesimal automorphisms. In particular, if $\nabla^{\mathcal{A}} s = 0$ for an adjoint Tractor, the above matrix splitting of $\nabla^{\mathcal{A}}$ shows that $\Pi(s) = \mathbf{k}_s$ is a conformal Killing field, and hence $L(\Pi(s)) = s$ satisfies

$$0 = \nabla^{\mathcal{A}} s = -K^\omega(\Pi(s), .).$$

Furthermore, we have the following "trace formula" for geometric objects associated with a parallel adjoint Tractor (cf. Lemma 2.5 of [19]:

Proposition 73 *Let $s \in \mathcal{A}(M)$ be a parallel adjoint Tractor and $\mathbf{k}_s = \Pi(s)$ its corresponding conformal Killing field. For any $g \in c$, let $\mathbf{K}_s = \frac{1}{2}\mathrm{rot}^g(\mathbf{k}_s)$ be the corresponding skew-symmetric endomorphism and let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis with respect to g. If $\mathbf{k}_s$ is Killing with respect to g (i.e. if $\mathrm{div}^g(\mathbf{k}_s) = 0$), then:*

$$\sum_{i=1}^{n} \epsilon_i K^\omega(\nabla^g_{e_i}\mathbf{k}_s, e_i) = \sum_{i=1}^{n} \epsilon_i K^\omega(\mathbf{K}_s(e_i), e_i) = 0. \tag{4.6}$$

Proof: First note that the first equality in (4.6) follows directly from the assumption that $\mathbf{k}_s$ is a conformal Killing field, which means

$$\nabla^g_{e_i}\mathbf{k}_s = \mathbf{K}_s(e_i) + \lambda \cdot e_i$$

for some function λ. Plugging this into the summands, the identity follows from skew-symmetry of K^ω.

From the assumptions, the one-form $\mathbf{k} \lrcorner K^\omega \in \Omega^1(M; \mathfrak{g})$ vanishes identically, and hence taking the divergence with respect to an orthonormal basis, we have:

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \epsilon_i e_i \lrcorner \nabla^g_{e_i}(\mathbf{k} \lrcorner K^\omega) \\
&= \sum_{i=1}^{n} \epsilon_i (\nabla^g_{e_i}(K^\omega(\mathbf{k}, e_i)) - K^\omega(\mathbf{k}, \nabla^g_{e_i} e_i)) \\
&= \sum_{i=1}^{n} \epsilon_i \nabla^g_{e_i}(K^\omega)(\mathbf{k}, e_i) + \sum_{i=1}^{n} \epsilon_i K^\omega(\nabla^g_{e_i}\mathbf{k}, e_i).
\end{aligned}$$

We claim that the first summand in the final line vanishes, from which the Proposition follows. To see this, recall the matrix forms of the curvature endomorphisms on standard and adjoint Tractors, respectively, given by the choice of metric $g \in c$ (recall, e.g. the formula (3.10) from Chapter 3.1):

$$\mathcal{R}^{\mathcal{T}}(X,Y) = \begin{pmatrix} 0 & C^g(X,Y)^* & 0 \\ 0 & W^g(X,Y) & -C^g(X,Y) \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathcal{R}^{\mathcal{A}}(X,Y) = \begin{pmatrix} W^g(X,Y) & 0 & 0 & 0 \\ -C^g(X,Y)^* \wedge & W^g(X,Y) & 0 & 0 \\ C^g(X,Y)^* & 0 & W^g(X,Y) & 0 \\ 0 & C^g(X,Y)^* \lrcorner & C^g(X,Y)^* \wedge & W^g(X,Y) \end{pmatrix}.$$

Here the Cotton-York tensor C^g is considered as a $(2,1)$-tensor, $C^g \in \Lambda^2(T^*M) \otimes TM$:

$$C^g(X,Y) := (\nabla^g_X \mathrm{P}^g)(Y) - (\nabla^g_Y \mathrm{P}^g)(X).$$

From $\mathcal{R}^{\mathcal{A}}(X,Y)s = 0$ and $\mathcal{R}^{\mathcal{T}}(\mathbf{k}_s, X) = 0$, we get the following identities:

$$W^g(X,Y,\mathbf{k}_s,Z) = 0; \tag{4.7}$$

$$g(C^g(\mathbf{k}_s, X), Y) = 0; \tag{4.8}$$

for arbitrary vectors $X, Y, Z \in TM$. From the Bianchi identity in semi-Riemannian geometry, on derives the following standard identity:

$$\sum_{i=1}^{n} \epsilon_i (\nabla_{e_i} W^g)(X,Y,Z,e_i) = (3-n) g(C^g(X,Y), Z).$$

Plugging in $\mathbf{k}_s$ for X in this equality, and using the indetities (4.8) and then (4.7), shows that the left-hand side vanishes, and it shows the following identity:

$$g(C^g(X,Y),\mathbf{k}_s) = 0. \tag{4.9}$$

Now we use the fact that $\mathbf{k}_s$ is Killing with respect to g, which in view of the Killing equation and the identities relating $\mathbf{K}_s$ and α_s to $\mathbf{k}_s$ for a parallel adjoint tractor, means that $\mathbf{K}_s = \nabla\mathbf{k}_s$ and $\alpha_s = 0$. Thus looking at the third row of the matrix (4.5), we see that s parallel implies $\gamma = -\mathrm{P}^g(\mathbf{k}_s, .)$. And looking at the second row of the same matrix implies

$$\nabla_X\nabla_Y\mathbf{k}_s = \mathrm{P}^g(X,Y)\mathbf{k}_s - \mathrm{P}^g(\mathbf{k}_s,Y)X \tag{4.10}$$
$$+ g(X,Y)\mathrm{P}^g(\mathbf{k}_s)^\sharp - g(\mathbf{k}_s,Y)\mathrm{P}^g(X)^\sharp. \tag{4.11}$$

Finally, again using the conformal version of the semi-Riemannian Bianchi identity, together with several applications of (4.7) and (4.9), gives:

$$\sum_{i=1}^n g(C^g(e_i,\nabla_{e_i}\mathbf{k}_s),Z) = -\frac{1}{n-3}\sum_{i,j=1}^n W^g(e_i,\nabla_{e_j}\nabla_{e_i}\mathbf{k}_s,e_j,Z) \tag{4.12}$$

Plugging (4.11) into (4.12) shows that the right-hand side of the latter identity vanishes, which completes the proof. $\square$

4.2.3 Geometry of the converse to the Fefferman constructions

Now we want to describe the converse to the Fefferman constructions geometrically, for both of the holonomy reductions $Hol(M,c) \subset SU(p'+1,q'+1) = H$ and $Hol(M,c) \subset Sp(p''+1,q''+1) = H'$, and use the results from the previous section to establish some properties. For the essential aspects in the case of $Hol(M,c) \subset H$, we follow the approach in Section 11 of [45].

From the conformal Holonomy reduction, we have parallel complex structures $J^{\mathcal{T}}$ or $J_i^{\mathcal{T}}$, $i = 1,2,3$ with the additional quaternionic commutation conditions, respectively. We identify these with $\nabla^{\mathcal{A}}$-parallel sections of the adjoint Tractor bundle satisfying the corresponding relations. The first step to a geometric description is given by the following Proposition describing the form of these adjoint Tractors with respect to an arbitrary metric $g \in c$:

Proposition 74 *Let $J^{\mathcal{T}} \in \Gamma(\mathcal{A})$, and let*

$$J^{\mathcal{T}} = \begin{pmatrix} -\alpha & \gamma & 0 \\ \mathbf{j} & J & -\gamma^\sharp \\ 0 & -\mathbf{j}^* & \alpha \end{pmatrix}$$

be the matrix decomposition of $J^{\mathcal{T}}$ with respect to a metric $g \in c$. Then $(J^{\mathcal{T}})^2 = -\mathrm{id}$ if and only if the following conditions are met by the vector fields $\mathbf{j}, -\gamma^\sharp \in \Gamma(TM)$ and the endomorphism $J \in \mathfrak{so}(TM)$:
(1) $\mathbf{j}$ and $-\gamma^\sharp$ are lightlike eigenvectors of J for the eigenfunction α;
(2) $g(\mathbf{j}, -\gamma^\sharp) = 1+\alpha^2$;
(3) J defines an almost complex structure on the codimension two distribution $\tilde{\mathcal{V}} \subset TM$ defined pointwise by $\tilde{\mathcal{V}}_x := (\mathbb{R}\mathbf{j}(x) \oplus \mathbb{R}\gamma^\sharp(x))^{\perp_g}$.

Proof: This is a reformulation of the Lemma 8 of [45], which is proved by squaring the matrix form of $J^{\mathcal{T}}$ and comparing the result with $-\mathrm{id}$.

The distribution spanned by $\mathbf{j}$ is trivially integrable, and we have projections $\psi : M \to N$ onto the local leaf space. The CR structure on N is essentially defined by taking

$$\mathcal{V} = \psi_*(\tilde{\mathcal{V}})$$

and projecting the almost complex structure J. Precisely, if we consider the subbundles

$$\mathbb{R}\mathbf{j} \subset (\mathbb{R}\mathbf{j})^{\perp_g} \subset TM,$$

then the tensorial endomorphism of TM defined by

$$X \mapsto \nabla^g_X \mathbf{j},$$

preserves both subbundles as a result of the properties discussed above. In particular, it descends to a tensorial map on the quotient bundle $(\mathbb{R}\mathbf{j})^{\perp_g}/(\mathbb{R}\mathbf{j})$, which we can see is identified with $\mathcal{V}$ and agrees with the projection of the almost complex structure J from $\tilde{\mathcal{V}}$.

It is also a straightforward application of the definitions and properties above, to see that both $\mathcal{V}$ and the induced complex structure on it, come from invariant subbundles of the Tractor bundle, its canonical linear connection and the parallel adjoint Tractor $J^{\mathcal{T}}$. In this way we see that the CR structure on N is invariant under choice of metric $g \in c$, which was shown in detail in [45], and corresponds to the regular infinitesimal flag structure induced by the Cartan connection in Proposition 70.

We note that the pseudo-Hermitian forms for this CR structure are described quite simply from the conformal manifold (M, c). Take $g \in c$ a metric for which $\mathbf{k}_s$ is Killing (whose existence is *a priori* only guaranteed locally). Then g is invariant under the local flows of $\mathbf{j}$, and $\tilde{\theta} := g(\gamma^\sharp, .)$ determines a well-defined one-form θ on N, whose kernel is precisely $\mathcal{V}$, i.e. a pseudo-Hermitian form for the CR manifold $(N, \mathcal{V}, J)$. It follows that the (symmetric) Levi form determined by θ corresponds to the metric on $\mathcal{V}$ induced by $g_{|\tilde{\mathcal{V}}}$, and so has (real) signature $(p-1, q-1)$. For the resulting pseudo-Hermitian forms we have:

Lemma 75 *Given a pseudo-Hermitian structure θ induced as above on the CR leaf space $(N, \mathcal{V}, J)$, and a unitary local basis $\{E_1, \ldots, E_{n'}\}$ with respect to L_θ, the "complex trace" of the Cartan curvature vanishes:*

$$\sum_{i=1}^{n'} K^{\omega'}(J(E_i), E_i) = 0, \tag{4.13}$$

for ω' the Cartan connection of type (H, Q) given by Proposition 70.

This is proved by an application of (4.6), which we shall do in detail when we prove the corresponding identity for the quaternionic case. First we describe the induced QC structure on the leaf space for a conformal holonomy reduction to $H' = Sp(p''+1, q''+1)$. In this case, we have three parallel complex structures $J_1^{\mathcal{T}}, J_2^{\mathcal{T}}, J_3^{\mathcal{T}}$ on the standard conformal Tractor bundle, and we denote the corresponding objects to those discussed above by the same symbols, with subscripts added.

Multiplying the matrix forms and comparing with the quaternionic commutation relations, we get:

$$\begin{aligned} J_i(\mathbf{j}_j) &= \mathbf{j}_k + \alpha_j \cdot \mathbf{j}_i; \\ J_i(\gamma_j^\sharp) &= \gamma_k^\sharp + \alpha_j \cdot \gamma_j^\sharp; \\ g(-\gamma_i^\sharp, \mathbf{j}_j) &= \alpha_k + \alpha_j \cdot \alpha_i = g(\mathbf{j}_i, \gamma_j^\sharp); \\ g(\mathbf{j}_r, \mathbf{j}_s) &= 0 = g(\gamma_r^\sharp, \gamma_s^\sharp); \\ g(\mathbf{j}_r, -\gamma_r^\sharp) &= 1 + (\alpha_r)^2; \end{aligned}$$

where (i, j, k) is any cyclic permutation of $(1, 2, 3)$ and $r, s = 1, 2, 3$ are arbitrary. We also have analogous identities, with signs reversed, for the cyclic permutations of $(1, 3, 2)$.

It is clear that the $\mathbf{j}_i$ and the $\gamma_i^\sharp$ are linearly independent and span a non-degenerate subbundle of TM, and we can define a co-dimension six distribution $\tilde{\mathcal{V}}'$ on M by

$$\tilde{\mathcal{V}}' := < \{\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3, \gamma_1^\sharp, \gamma_2^\sharp, \gamma_3^\sharp\} >^{\perp_g} .$$

This is evidently equivalent to

$$\tilde{\mathcal{V}}' = \bigcap_{i=1}^{3} \tilde{\mathcal{V}}_i$$

for $\tilde{\mathcal{V}}_i$ the codimension two distribution associated to each of the $J_i^{\mathcal{T}}$ as above. In particular, J_1, J_2, J_3 are three almost complex structures on $\tilde{\mathcal{V}}'$ which fulfill the commutation relations of the quaternions.

We have the integrable distribution $\mathcal{V}^{\mathfrak{q}'}$ spanned by $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$, and projections $\psi' : M \to N'$ onto a local leaf space, and in addition three different projections $\psi_i : M \to N_i$ onto a local leaf space for each of the $\mathbf{j}_i$, and these in turn project locally onto N', which we denote $\psi_i' : N_i \to N$. The induced QC structure on N' is defined as follows: Let $\mathcal{V}' := \psi_*'(\tilde{\mathcal{V}}')$. This is a corank three distribution, which has a fixed conformal class of metrics on it, induced by the conformal class on M, restricted to $\tilde{\mathcal{V}}'$. The conformal class of metrics is well-defined, since $\mathcal{V}^{\mathfrak{q}'}$ is spanned by conformal Killing fields.

The bundle of complex structures on $\mathcal{V}'$ is given by taking local sections of any of the three projections $\psi_i' : N_i \to N'$. For example, picking a local scale $g \in c$ with respect to which $\mathbf{j}_1$ is Killing, we get a pseudo-Hermitian form θ_1 on the CR manifold $(N_1, \mathcal{V}_1, J_1)$. In addition, let

$$\theta_2 := L_\theta((\psi_1)_*(\mathbf{j}_2), .), \theta_3 := L_\theta((\psi_1)_*(\mathbf{j}_3), .).$$

Then for a local section $\sigma : U \to N_1$ of the projection ψ_1', a local, $\mathbb{R}^3$-valued one-form on N', $\theta^\sigma \in \Omega^1(U, \mathbb{R}^3)$, is defined by

$$\theta^\sigma = (\theta_1^\sigma, \theta_2^\sigma, \theta_3^\sigma) = (\sigma^*(\theta_1), \sigma^*(\theta_2), \sigma^*(\theta_3)).$$

The requirements for the local contact form of a QC structure (cf. Definition 66 in Chapter 4.1) are straightforward to verify, and this shows that the distribution $\mathcal{V}'$ on N' has a $CSp(1)Sp(p'', q'')$ structure.

Applying the trace formula (4.6) also gives a quaternionic version for the QC structure on N':

Lemma 76 *Let I_1, I_2, I_3 be a quaternionic triple which locally generating the bundle of complex structures on the QC distribution $\mathcal{V}'$ defined as above. Then for each $r = 1, 2, 3$, there exists a local QC contact form θ^{σ_r} such that, for any unitary basis $\{E_1, \ldots, E_{n''}\}$ of $\mathcal{V}'$ with respect to I_r and the induced metric g_r, the following identity holds:*

$$\sum_{i=1}^{n''} \epsilon_i K^{\omega''}(I_r(E_i), E_i) = 0. \tag{4.14}$$

Proof: From the definition of the local trivialization of the bundle of complex structures on $\mathcal{V}'$, it follows that there exists a local section $\sigma_r : U \to N_r$ of the projection map $\psi'_r : N_r \to N'$ such that $\sigma_r^*(\theta_r)$ determines the complex structure I_r on $\mathcal{V}'$ by the compatibility condition (4.2):

$$g_r(X, Y) = d(\theta_r^{\sigma_r})(X, I_r Y).$$

Here, θ_r is the pseudo-Hermitian form on the CR manifold N_r, defined by choosing a (local) scale $g \in c$ in the conformal class of metrics on M such that $\mathbf{j}_r$ is Killing, and g_r is the locally defined metric on $\mathcal{V}'$ given by pulling back the Levi form L_{θ_r} by σ_r.

Then evidently the formula (4.14) can be verified for the local QC contact form defined by σ_r, by checking the corresponding identity on N_r, and this in turn can be lifted to the conformal manifold M. The curvature tensors $K^{\omega''}, K^{\omega'}$ and K^{ω}, correspond under these changes, so this makes sense. Then we can apply (4.6) for the parallel adjoint Tractor $J_r^{\mathcal{T}}$ and the Killing scale $g \in c$ for the corresponding conformal Killing field $\mathbf{j}_r$, for a particular choice of local orthonormal basis of TM with respect to g.

To get this basis, we can lift any unitary basis of $\mathcal{V}'$ for I_r to M and

$$\{E_1, J_r(E_1), \ldots, E_{n''}, J_r(E_{n''})\}$$

gives an orthonormal basis of $\tilde{\mathcal{V}}'$. The appropriate orthonormal basis of the orthogonal complement of $\tilde{\mathcal{V}}'$ is defined as follows, for (r, s, t) a cyclic permutation of $(1, 2, 3)$: By definition, $\alpha_r = 0$ for the chosen scale, and hence $g(\mathbf{j}_r, \gamma_r^\sharp) = -1$. Let

$$e_r^+ := \frac{1}{\sqrt{2}}(\mathbf{j}_r - \gamma_r^\sharp);$$
$$e_r^- := \frac{1}{\sqrt{2}}(\mathbf{j}_r + \gamma_r^\sharp).$$

Then e_r^+ has length 1, e_r^- has length -1, and they are mutually orthogonal. Now, if either $\mathbf{j}_s$ or $\mathbf{j}_t$ is Killing with respect to g, then one can see that this must automatically hold for the third of our conformal Killing fields. This case is very simple to check, so we assume in the following that it doesn't occur, and in particular, α_s, α_t are both non-zero. Then let

$$\mathbf{j}_{s,t} := \frac{1}{\sqrt{c_{s,t}}}(\frac{1}{\alpha_t}\mathbf{j}_s + \frac{1}{\alpha_s}\mathbf{j}_t);$$
$$\gamma_{s,t}^\sharp := \frac{1}{\sqrt{c_{s,t}}}(\frac{1}{\alpha_t}\gamma_s^\sharp + \frac{1}{\alpha_s}\gamma_t^\sharp);$$

where $c_{s,t}$ is a positive constant, which is minus the pairing of the vectors in parentheses on the two lines. Explicitly,

$$c_{s,t} = \frac{(\alpha_s)^2(1 + (\alpha_s)^2) + (\alpha_t)^2(1 + (\alpha_t)^2)}{(\alpha_s)^2(\alpha_t)^2}.$$

Note that $\mathbf{j}_{s,t}$ and $\gamma^\sharp_{s,t}$ are light-like vectors, that $g(\mathbf{j}_{s,t}, \gamma^\sharp_{s,t}) = -1$, and from the definition both lie in the orthogonal complement $(\mathbb{R}\mathbf{j}_r \oplus \mathbb{R}\gamma^\sharp_r)^\perp$. In particular, they belong to the distribution on which the endomorphism J_r defines an almost complex structure. Finally, we define

$$e^+_{s,t} := \frac{1}{\sqrt{2}}(\mathbf{j}_{s,t} - \gamma^\sharp_{s,t});$$
$$e^-_{s,t} := \frac{1}{\sqrt{2}}(\mathbf{j}_{s,t} + \gamma^\sharp_{s,t});$$

and get an orthonormal local basis of TM with respect to g, given by

$$\{e^+_r, e^-_r, e^+_{s,t}, J_r(e^+_{s,t}), e^-_{s,t}, J_r(e^-_{s,t}), E_1, J_r(E_1), \ldots, E_{n''}, J_r(E_{n''})\}.$$

Applying (4.6) for this basis, we get

$$\begin{aligned} 0 =& K^\omega(J_r(e^+_r), e^+_r) + K^\omega(J_r(e^-_r), e^-_r) \\ &+ 2K^\omega(J_r(e^+_{s,t}), e^+_{s,t}) - 2K^\omega(J_r(e^-_{s,t}), e^-_{s,t}) \\ &+ 2\sum_{i=1}^{n''} \epsilon_i K^\omega(J_r(E_i), E_i). \end{aligned}$$

Now the Lemma follows by showing that the sum of the first four terms on the right hand side vanishes. The first two terms vanish identically, because e^+_r and e^-_r are the sum of vectors which are eigenvectors of 0 for the endomorphism J_r. For the third and fourth summands, note that $\mathbf{j}_{s,t}$ and $J_r(\mathbf{j}_{s,t})$ both insert trivially into the curvature form K^ω, as a result of the properties of parallel adjoint Tractors. But applying this, and the definitions of $e^+_{s,t}$ and $e^-_{s,t}$, we see that the remaining, non-zero, terms in the third and fourth summands cancel. □

4.3 Symplectic conformal holonomy

Propositions 69 and 70 give, using only rather general features of the groups H and H', and the associated geometry, a partial holonomy correspondence. To prove the conformal holonomy correspondence for the symplectic group $H' := Sp(p''+1, q''+1)$ given in Theorem 1, it remains to establish normality in both directions:

Proposition 77 *The induced Cartan connection ω of type (G, P) in the second part of Proposition 69 is always normal (it is automatically regular, since $\mathfrak{g}$ is $|1|$-graded). The induced Cartan connection ω'' of type (H', Q') in the second part of Proposition 70 is normal and torsion-free (and therefore regular).*

This Section is devoted to a proof of this Proposition. In the course of doing this, most of what's needed for the holonomy correspondence for integrable CR structures and the special unitary group H (cf. [18] and [19]) is also established along the way. To begin with, these properties may all be established in a purely local context. So by the previous Section we may take as our context a conformal Fefferman space $(\mathcal{P}, \pi, M, \omega)$ of type (G, P) (not necessarily the canonical Cartan geometry for the conformal structure on M), defined over some parabolic geometry $(\mathcal{Q}', \pi'', N', \omega'')$ of type (H', Q'), and we also have a parabolic geometry of type (H, Q) between the two.

Recalling the commutative diagram from Proposition 56 in Chapter 3.3, which relates the curvature functions $\kappa, \tilde{\kappa}, \kappa'$, etc., we see that restricting to $u' \in \mathcal{P}^{H'}$, we may view the element $\kappa(u')$ either as an element of $C^2(\mathfrak{g}_-, \mathfrak{g})$, of $C^2(\mathfrak{h}_-, \mathfrak{h})$, or of $C^2(\mathfrak{h}'_-, \mathfrak{h}')$, and it corresponds in this way to the curvature functions κ' and κ'', respectively. By the Ad-equivariance of the normality condition, though, it is no problem to restrict to $\mathcal{P}^{H'}$ to check normality. For notational convenience, we will also often omit the point u', and simply write κ as an element of the various chain groups. We will write $\partial^*_{\mathfrak{p}} \circ \kappa$, $\partial^*_{\mathfrak{q}} \circ \kappa$, or $\partial^*_{\mathfrak{q}} \circ \kappa$ for the co-differential operators corresponding to the different graded Lie algebras applied to κ, according to whether we consider κ as an element of $C^2(\mathfrak{g}_-, \mathfrak{g})$, $C^2(\mathfrak{h}_-, \mathfrak{h})$ or $C^2(\mathfrak{h}'_-, \mathfrak{h}')$, respectively. Then Proposition 77 can thus be reformulated as:

Proposition 78 *If $\partial^*_{\mathfrak{q}'} \circ \kappa = 0$ and κ is torsion-free of type (H', Q'), then $\partial^*_{\mathfrak{p}} \circ \kappa = 0$. If $\partial^*_{\mathfrak{p}} \circ \kappa = 0$, then $\partial^*_{\mathfrak{q}'} \circ \kappa = 0$, and κ is automatically torsion-free of type (H', Q').*

Note first of all that the torsion condition follows directly from the diagram in Proposition 56, since a Cartan connection of conformal type (G, P) is normal only if it is torsion-free. The diagram shows that κ is torsion-free of type (G, P), i.e. $\kappa(X, Y) \in \mathfrak{p}$ for all $X, Y \in \mathfrak{g}$ only if it is torsion-free of types (H, Q) and (H', Q'), respectively. On the other hand, as was discussed at the end of Chapter 4.1.2, the canonical Cartan connection for a QC^+ structure is always torsion-free of type (H', Q') (we assume integrability in dimension seven).

4.3.1 Algebraic identities

Proving the proposition reduces to a series of calculations with Lie algebras. Recall the formula (2.29) from Proposition 16 in Chapter 2.2, which gives an explicit formula for each of the codifferential operators we're considering in terms of dual bases of $\mathfrak{g}_-$ and $\mathfrak{g}_+$, of $\mathfrak{h}_-$ and $\mathfrak{h}_+$, or of $\mathfrak{h}'_-$ and $\mathfrak{h}'_+$, respectively. In particular, the formulae (2.30) and (2.31) define any of the codifferentials as the sum of two operators:

$$\partial^*_{\mathfrak{f}} \circ \phi = (\partial^*_{\mathfrak{f}} \circ \phi)_1 - (\partial^*_{\mathfrak{f}} \circ \phi)_2,$$

where we note that for $\mathfrak{f} = \mathfrak{p}$, the second operator in this sum automatically vanishes, since $\mathfrak{g}$ is $|1|$-graded. We claim the following identities for these operators:

Lemma 79 *For the curvature function κ as in the assumptions of Proposition 78, the following algebraic relations hold for the operators $(\partial^*)_1$ and $(\partial^*)_2$:*

$$\frac{1}{2}(\partial^*_{\mathfrak{q}}\kappa)_1 = \mathrm{pr}_{\mathfrak{h}}(\partial^*_{\mathfrak{p}}\kappa); \tag{4.15}$$

$$\frac{1}{2}(\partial^*_{\mathfrak{q}'}\kappa)_1 = \mathrm{pr}_{\mathfrak{h}'}((\partial^*_{\mathfrak{q}}\kappa)_1); \tag{4.16}$$

$$(\partial^*_{\mathfrak{q}'}\kappa)_2(\mathfrak{h}'_{-2}(i)) = (\partial^*_{\mathfrak{q}}\kappa)_2(\mathfrak{h}_{-2}(i)). \tag{4.17}$$

Proof: – Identity (4.15): First we choose special bases with which to compute the terms on either side of the identity. Let $n' = p' + q'$. Then $\mathfrak{h}_- \cong \mathbb{R} \oplus \mathbb{C}^{n'}$, and we fix the following real basis: $X_\alpha := \mathfrak{h}_{-1}(e_\alpha)$ and $X_{\alpha+n'} := \mathfrak{h}_{-1}(ie_\alpha)$ for $\alpha = 1, \ldots n'$ and $\{e_1, \ldots e'_n\}$ the standard pseudo-orthonormal basis of $\mathbb{R}^{p',q'}$; and $X_0 = X_I := \mathfrak{h}_{-2}(i)$. Now let

$$\mathfrak{su}(p'+1, q'+1) \overset{\varphi}{\hookrightarrow} \mathfrak{so}(p+1, q+1)$$

$$A + iB \overset{\varphi}{\longmapsto} \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

be the standard inclusion. This gives the matrix form of a map $X \in \mathfrak{su}(p'+q, q'+q)$ with respect to the real ordered basis $\{e_0, \ldots, e_{n'}, ie_0, \ldots, ie_{n'}\}$. Then denote by ψ the automorphism od $\mathfrak{so}(p+1, q+1)$ associating to changing this basis to the ordered basis $\{f_0, \ldots, f_{2n'+3}\}$ defined by:

$$f_0 := e_0; \quad f_{2n'+3} := e_{n'+1};$$
$$f_1 := \frac{1}{\sqrt{2}}(ie_0 - ie_{n'+1}); \quad f_{2n'+2} := \frac{1}{\sqrt{2}}(ie_0 + ie_{n'+1});$$
$$f_{2\alpha} := e_\alpha; \quad f_{2\alpha+1} := ie_\alpha, \quad \forall\, 1 \leq \alpha \leq n'.$$

We define a grading of the inclusion $\varphi : \mathfrak{su}(p'+1, q'+1) \hookrightarrow \mathfrak{so}(p+1, q+1)$ by letting $\varphi_\bullet = (\varphi_{-1}, \varphi_0, \varphi{+}1)$, for

$$\varphi_i := \mathrm{pr}_{\mathfrak{g}_i} \circ \psi \circ \varphi$$

for $i = -1, 0, +1$. Then for our standard basis elements of $\mathfrak{h}$ from above, it is straightforward to check the following identities:

$$\psi \circ \varphi(X_0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ (f_{2n'+2} - f_1) & 0 & 0 \\ 0 & (f_1 - f_{2n'+2})^{\psi t} & 0 \end{pmatrix} \tag{4.18}$$
$$=: \varphi_{-1}(X_0); \tag{4.19}$$

$$\psi \circ \varphi(X_\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ f_{2\alpha} & 0 & 0 \\ 0 & -f_{2\alpha}^{\psi t} & 0 \end{pmatrix} + \varphi_0(X_\alpha) \tag{4.20}$$
$$=: \varphi_{-1}(X_\alpha) + \varphi_0(X_\alpha); \tag{4.21}$$

$$\psi \circ \varphi(X_{\alpha+n'}) = \begin{pmatrix} 0 & 0 & 0 \\ f_{2\alpha+1} & 0 & 0 \\ 0 & -f_{2\alpha+1}^{\psi t} & 0 \end{pmatrix} + \varphi_0(X_{\alpha+n'}) \tag{4.22}$$
$$=: \varphi_{-1}(X_{2\alpha+n'}) + \varphi_0(X_{\alpha+n'}); \tag{4.23}$$

and for $l = 0, \ldots, 2n'$ and $Z_l \in \mathfrak{h}_+$ the transpose of X_l, we have

$$\psi \circ \varphi(Z_l) = \varphi_0(Z_l) + \varphi_{+1}(Z_l)$$

with $\varphi_i(Z_l) = (\varphi_{-i}(X_l))^t$. In particular, note that $(\varphi_{+1})_{|\mathfrak{h}_-} \equiv 0$ and $(\varphi_{-1})_{|\mathfrak{h}_+} \equiv 0$. Finally, for any element $\delta \in \mathfrak{h}_0$ of the form

$$\delta = \begin{pmatrix} a+i & 0 & 0 \\ 0 & A_0 + iB_0 & 0 \\ 0 & 0 & -a+i \end{pmatrix},$$

we have the identity:

$$\psi \circ \varphi(\delta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ (f_1 + f_{2n'+2}) & 0 & 0 \\ 0 & (f_1 + f_{2n'+2})^{\psi t} & 0 \end{pmatrix} \tag{4.24}$$
$$+ \varphi_0(\delta) + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & (f_1 - f_{2n'+2})^{\psi t} & 0 \\ 0 & 0 & (f_{2n'+1} - f_1) \\ 0 & 0 & 0 \end{pmatrix} \tag{4.25}$$
$$=: \varphi_{-1}(\delta) + \varphi_0(\delta) + \varphi_{+1}(\delta). \tag{4.26}$$

We use the above formulae to define a real basis $\{\tilde{X}_l\}$ of $\mathfrak{g}_-$, letting $\tilde{X}_l := \varphi_{-1}(X_l)$ for $0 \le l \le 2n'$, and taking $\tilde{X}_{2n'+2} := \varphi_{-1}(\delta)$ for $\delta \in \mathfrak{h}_0$ of the above form.

To fix a dual basis of $\mathfrak{g}_+$, we use a constant multiple of the Killing form which is more convenient for explicit calculations and does not affect normality, since it merely changes the terms we are computing by a constant multiple. Let the bilinear form B on $\mathfrak{gl}(n)$ be given by $B(X,Y) := \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(X \circ Y)$. This is a non-degenerate, Ad-invariant real-valued form on the Lie algebras we're considering, and it's a constant multiple of the Killing form for each of the Lie algebras $\mathfrak{g}, \mathfrak{h}, \mathfrak{h}'$ we're considering.

Now let $\{\tilde{Z}_l\}$ be the basis of $\mathfrak{g}_+$ which is dual to $\{\tilde{X}_l\}$ with respect to B, i.e. $B(\tilde{X}_l, \tilde{Z}_k) = \delta_{lk}$. Letting $\{Z_k\}$, $0 \le k \le 2n'$ be the basis of $\mathfrak{h}_+$ given by taking Z_k to be the pseudo-hermitian transpose to X_k with respect to the form $I^{1,1}_{p',q'}$, it follows that $\{Z_k\}$ is dual to $\{X_k\}$ with respect to B, and the above identities show that:

$$\varphi_{+1}(Z_j) = \tilde{Z}_j, \ \forall 1 \le j \le 2n';$$
$$\varphi_{+1}(Z_0) = 2\tilde{Z}_0.$$

In particular, let us denote $\tilde{Z}_l^{\mathfrak{h}} := \mathrm{pr}_{\mathfrak{h}}(\tilde{Z}_l)$ and $\tilde{Z}_l^{\perp} := \mathrm{pr}_{\mathfrak{h}^\perp}(\tilde{Z}_l)$, where $pr_{\mathfrak{h}}$ and $pr_{\mathfrak{h}^\perp}$ are the projections given by the orthogonal splitting $\mathfrak{g} = \varphi(\mathfrak{h}) \oplus \varphi(\mathfrak{h})^\perp$. Then we have, for $l = 1, \ldots, 2n$:

$$\tilde{Z}_l = \tilde{Z}_l^{\mathfrak{h}} + \tilde{Z}_l^{\perp} = \frac{1}{2}(\varphi_1(Z_l) + \varphi_0(Z_l)) + \frac{1}{2}(\varphi_1(Z_l) - \varphi_0(Z_l)); \tag{4.27}$$
$$\tilde{Z}_0 = \tilde{Z}_0^{\mathfrak{h}} = \frac{1}{2}\varphi(Z_0). \tag{4.28}$$

Now we may calculate, for any $X \in \mathfrak{g}$ (without loss of generality, $X \in \mathfrak{h}$):

$$(\partial^*_{\mathfrak{p}}\kappa)(X) = \sum_{l=0}^{2n+1}[\kappa(X, \tilde{X}_l), \tilde{Z}_l] \tag{4.29}$$
$$= \sum_{l=0}^{2n}[\kappa(X, \varphi(X_l)), \tilde{Z}_l^{\mathfrak{h}}] + \sum_{l=0}^{2n}[\kappa(X, \varphi(X_l)), \tilde{Z}_l^{\perp}] \tag{4.30}$$
$$= \frac{1}{2}(\partial^*_{\mathfrak{q}}\kappa)_1(X) + \sum_{l=0}^{2n}[\kappa(X, \varphi(X_l)), \tilde{Z}_l^{\perp} \tag{4.31}$$

Here the equality of lines (4.29) and (4.30) uses the fact that $\kappa(\tilde{X}_{2n+1}, .) = 0$ in our case, and that $\kappa(\varphi_0(X_l), .) = 0$, while (4.31) follows from (4.30) by applying the definition of $(\partial^*_{\mathfrak{q}}\kappa)_1$ and the fact that $\tilde{Z}_l^{\mathfrak{h}} = \frac{1}{2}\varphi(Z_l)$ as a result of (4.27) and (4.28). Now, the final term in (4.31) is contained in the orthogonal subspace $\varphi(\mathfrak{h})^\perp \subset \mathfrak{g}$, because $\kappa(X,Y) \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$, together with Ad-invariance of the Killing form. This proves identity (4.15) of Lemma 79.

Identity (4.16): This is established by a completely analogous calculation, using the standard embedding:

$$\mathfrak{sp}(p''+1, q''+1) \overset{\varphi'}{\hookrightarrow} \mathfrak{su}(p'+1, q'+1)$$
$$A + iB + jC + kD \overset{\varphi'}{\longmapsto} \begin{pmatrix} A + iB & -C - iD \\ C - iD & A - iB \end{pmatrix},$$

in order to construct real bases of $\mathfrak{h}'_-$ and $\mathfrak{h}_-$ and dual bases of $\mathfrak{h}'_+$ and $\mathfrak{h}_+$ in order to compare $(\partial^*_{\mathfrak{q}}\kappa)_1$ and $(\partial^*_{\mathfrak{q}'}\kappa)_1$. We will also apply the analogous notation as introduced for the calculation of (a), denoting by $\varphi'_\bullet = (\varphi'_{-2}, \dots, \varphi'_2)$ the graded inclusion defined in the same way as for the inclusion $\varphi : \mathfrak{su}(p'+1, q'+1) \hookrightarrow \mathfrak{so}(p+1, q+1)$.

Identity (4.17): This is a calculation with matrices, using the inclusion $\varphi'_\bullet$. We need to look at the form of a few of the basis vectors of $\mathfrak{h}'_-$ and their images under the graded inclusion $\varphi'_\bullet$. For this inclusion, the calculations are just like for $\varphi_\bullet$ above. If we let $\psi' : \mathfrak{su}(p'+1, q'+1) \to \mathfrak{su}(p'+1, q'+1)$ be the analogous automorphism of ψ, corresponding to changing the complex basis $\{z_0, \dots, z_{n''+1}, jz_0, \dots, jz_{n''+1}\}$ to a basis $\{u_0, \dots, u_{2n''+3}\}$, with respect to which the hermitian metric $<,>^{\mathbb{C}}$ is given by the form $I^{1,1}_{p',q'}$, then we have:

$$\begin{aligned}\psi' \circ \varphi'(\mathfrak{h}'_{-2}(i)) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} + \varphi'_0(\mathfrak{h}'_{-2}(i)) \\ &=: \varphi'_{-2}(\mathfrak{h}'_{-2}(i) + \varphi'_0(\mathfrak{h}'_{-2}(i)) = \mathfrak{h}_{-2}(i) + \varphi'_0(\mathfrak{h}'_{-2}(i)).\end{aligned}$$

On the other hand, the other two basis elements of $\mathfrak{h}'_{-2}$ are sent to basis elements of $\mathfrak{h}_{-1}$:

$$\begin{aligned}\psi' \circ \varphi'(\mathfrak{h}'_{-2}(j)) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ (u_{2n''+2} - u_1) & 0 & 0 \\ 0 & (u_1 - u_{2n''+2})^{\psi *} & 0 \end{pmatrix} \\ &=: \varphi'_{-1}(\mathfrak{h}'_{-2}(j));\end{aligned}$$

$$\begin{aligned}\psi' \circ \varphi'(\mathfrak{h}'_{-2}(k)) &= \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ (u_{2n''+2} - u_1) & 0 & 0 \\ 0 & (u_1 - u_{2n''+2})^{\psi *} & 0 \end{pmatrix} \\ &=: \varphi'_{-1}(\mathfrak{h}'_{-2}(k)).\end{aligned}$$

We have $\{\tilde{X}'_l\}_{l=-2}^{4n+3}$ and $\{\tilde{Z}'_l\}$, dual bases of $\mathfrak{h}_-$ and $\mathfrak{h}_+$ constructed from a basis $\{\tilde{X}'_l\}_{l=-2}^{4n''}$ of $\mathfrak{h}'_-$ by a process analogous to that used in the proof of Lemma 79. Using these:

$$(\partial^*_{\mathfrak{q}}\kappa)_2(\mathfrak{h}_{-2}(i)) = \sum_{l=-2}^{4n+3} \kappa([\mathfrak{h}_{-2}(i), \tilde{Z}'_l], \tilde{X}'_l) = \sum_{l=-2}^{4n} \kappa([\mathfrak{h}_{-2}(i), \tilde{Z}'_l], \tilde{X}'_l) \tag{4.32}$$

$$= \sum_{l=1}^{4n} \kappa([\varphi'(\mathfrak{h}'_{-2}(i)), \varphi'(Z'_l)], \varphi'(X'_l)) - \sum_{l=-2}^{4n} \kappa([\mathfrak{h}_{-2}(i), \varphi'_0(Z'_l)], \varphi'(X'_l)) \tag{4.33}$$

Here we have applied several times the fact that $\kappa(X, .)$ vanishes whenever $X \in \mathfrak{q} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ or $X \in \varphi'(\mathfrak{q}')$, in order to go from line (4.32) to line (4.33). Now we claim that the last term in (4.33) vanishes, and thus the identity (4.17) is proven. This claim follows since $[\mathfrak{h}_{-2}(i), \varphi'_0(Z'_l)]$ vanishes for all $l = -2, \dots, 4n$, as a result of writing $\mathfrak{h}_{-2}(i) = \psi' \circ \varphi'(\mathfrak{h}'_{-2}(i)) - \varphi'_0(\mathfrak{h}'_{-2}(i))$ and $\varphi'_0(Z'_l) = \psi' \circ \varphi'(Z'_l)$. Then similar to the case for the grading of $\mathfrak{su}(p'+1, q'+1)$, we have the identity

$$[\mathfrak{h}'_{-2}(i), \mathfrak{h}'_{+1}(v)] = -\mathfrak{h}_{-1}(iv)$$

for $v \in \mathbb{H}^{p'',q''} \cong \mathfrak{h}'_{-1}$. In particular, $[\psi' \circ \varphi'(\mathfrak{h}'_{-2}(i)), \psi' \circ \varphi'(Z'_l)] = \psi' \circ \varphi'(\mathfrak{h}'_{-1}(iv))$ for some $v \in \mathbb{H}^{p'',q''}$. Now, comparing the grading components, we see that it can't have a non-zero term in $\mathfrak{h}_{-2}$, and hence the commutator in question must vanish. □

4.3.2 Proof of Theorem 1

We first prove that a normal structure "upstairs" induces a normal structure "downstairs":

Lemma 80 *For κ as in the assumptions of Proposition 78, if $\partial^*_{\mathfrak{p}}\kappa = 0$, then $\partial^*_{\mathfrak{q}}\kappa = (\partial^*_{\mathfrak{q}}\kappa)_1 = 0$. Moreover, this also implies that $\partial^*_{\mathfrak{q}'}\kappa = (\partial^*_{\mathfrak{q}'}\kappa)_1 = 0$.*

Proof: From part (a) of Lemma 79, we know that $(\partial^*_{\mathfrak{q}}\kappa)_1 = \mathrm{pr}_{\mathfrak{h}}(\partial^*_{\mathfrak{p}}\kappa) = 0$ under the assumptions. Thus, to prove the first statement of this Lemma, it suffices to show that

$$(\partial^*_{\mathfrak{q}}\kappa)_2(X) = \frac{1}{2}\sum_{l=0}^{2n}\kappa([X, Z_l], X_l)$$

vanishes for all $X \in \mathfrak{h}_-$. Since $\kappa(Y, .) = 0$ for all $Y \in \mathfrak{q}$, it suffices to check this fact for $X \in \mathfrak{h}_{-2}$ and restrict to basis elements $Z_l \in \mathfrak{h}_1$, i.e. $l = 1, \ldots, 2n$. Note that for our basis element $X_0 = \mathfrak{h}_{-2}(i) \in \mathfrak{h}_{-2}$ and $v \in \mathbb{C}^{p',q'}$, simple matrix calculation yields: $[\mathfrak{h}_{-2}(i), \mathfrak{h}_1(v)] = -\mathfrak{h}_{-1}(iv)$. Thus

$$(\partial^*_{\mathfrak{q}}\kappa)_2(X_0) = \frac{1}{2}\sum_{l=1}^{2n}\kappa([\mathfrak{h}_{-2}(i), Z_l], X_l) \tag{4.34}$$

$$= \frac{1}{2}\sum_{\alpha=1}^{n}(\kappa(-\mathfrak{h}_{-1}(iX_\alpha), \mathfrak{h}_{-1}(X_\alpha)) + \kappa(-\mathfrak{h}_{-1}(-X_\alpha), \mathfrak{h}_{-1}(iX_\alpha))). \tag{4.35}$$

Now, from the pointwise isomorphism $\mathrm{gr}(T_xN) \cong \mathfrak{h}_-$ given by the partially integrable CR structure $(N, \mathcal{V}, J)$ defined on the local leaf space N at the end of the previous Section, we can identify, for $u \in \mathcal{P}^H$, and $\tilde{\xi}, \tilde{\eta} \in \omega(u)^{-1}(\mathfrak{h}_{-1}) \subset T_u\mathcal{P}^H$ and $\pi'_*(\tilde{\xi}) = \xi, \pi'_*(\tilde{\eta}) = \eta \in \mathcal{V}_{\pi'(u)}$, the curvature terms

$$\kappa(\omega(u)(\tilde{\xi}), i\omega(u)(\tilde{\eta})) = K(\xi, J\eta).$$

and by the identity (4.13) from the previous Section, we see that this term vanishes by choosing (if necessary, locally) a scale $g \in c$ for which $\mathbf{j}$ is Killing. I.e. for a local (unitary) basis $\{Y_1, \ldots Y_{2n'}\}$ of $\mathcal{V}$, we have:

$$\sum_{l=1}^{2n'} K(Y_l, JY_l) = 0.$$

Thus, $(\partial^*_{\mathfrak{q}}\kappa)_2(X) = 0$ for all $X \in \mathfrak{h}_-$. From the second identity of Lemma 79, now, we also have $(\partial^*_{\mathfrak{q}'}\kappa)_1 = \mathrm{pr}_{\mathfrak{h}'}((\partial^*_{\mathfrak{q}}\kappa)_1) = 0$, so to complete the proof of the present Lemma, it suffices to show that $(\partial^*_{\mathfrak{q}'}\kappa)_2$ vanishes. Again, it suffices to check this on basis elements of $\mathfrak{h}'_{-2}$. By an argument completely analogous to that above, this follows from the formula (4.14) using the commutator relations for the quaternionic Heisenberg algebra. □

Now, to establish Proposition 78, and therefore the conformal holonomy correspondence in Theorem 1 , it only remains to prove:

Proposition 81 *For κ the curvature operator as in the assumptions of Proposition 78, if κ is torsion-free considered as type (H', Q') and $\partial^*_{\mathfrak{q}'} \circ \kappa = 0$, then $\partial^*_{\mathfrak{p}} \circ \kappa = 0$.*

This proposition is established via the following two Lemmas:

Lemma 82 *If κ is regular and normal of type (H', Q'), then $(\partial^*_{\mathfrak{q}'}\kappa)_1 = (\partial^*_{\mathfrak{q}'}\kappa)_2 = 0$ and κ is torsion-free when considered as type (G, P) (hence also when considered as type (H, Q)).*

Lemma 83 *If $(\partial^*_{\mathfrak{q}'}\kappa)_1 = (\partial^*_{\mathfrak{q}'}\kappa)_2 = 0$ and κ is torsion-free of type (G, P), then $(\partial^*_{\mathfrak{q}}\kappa)_1 = (\partial^*_{\mathfrak{q}}\kappa)_2 = 0$. Moreover, this implies that $\partial^*_{\mathfrak{p}} \circ \kappa = 0$.*

Proof of Lemma 82: The first statement in this Lemma is in fact a general feature of ***all*** torsion-free, normal parabolic Cartan connections, cf. the proof of statement (1) in Theorem 3.5 of [15]. First, using the identification $\mathfrak{h}'_+ \cong (\mathfrak{h}'_-)^*$, we can rewrite any 2-chain $\beta \in C^2(\mathfrak{h}'_-, \mathfrak{h}')$ as a sum

$$\sum_{1 \leq i < j \leq 4n''+3} Z_i \wedge Z_j \otimes t_{ij};$$

where $\{Z_1, \ldots, Z_{4n''+3}\}$ is a basis of $\mathfrak{h}'_+$ and $t_{ij} \in \mathfrak{h}'$. Then one sees that the codifferential $\partial^*_{\mathfrak{q}'}$ can be reformulated (on basis vectors) as

$$\partial^*_{\mathfrak{q}'} : Z_i \wedge Z_j \otimes t_{ij} \mapsto Z_i \otimes [Z_j, t_{ij}] - Z_j \otimes [Z_i, t_i j] - [Z_i, Z_j] \otimes t_{ij},$$

and we see that the operator $(\partial^*_{\mathfrak{q}'})_2$ corresponds to the map

$$[\,,] \otimes \mathrm{id} : \Lambda^2(\mathfrak{h}'_+) \otimes \mathfrak{h}' \to \Lambda^1(\mathfrak{h}'_+) \otimes \mathfrak{h}'.$$

The irreducible component $H^2_2(\mathfrak{h}'_-, \mathfrak{h}')$ (in which κ_H lives) corresponds to a $\mathfrak{h}'_0$-submodule in $\Lambda^2(\mathfrak{h}'_+) \otimes \mathfrak{h}'$. The map $[\,,] \otimes \mathrm{id}$ gives a homomorphism of $\mathfrak{h}'_0$-submodules, so by Schur's Lemma it is either identically zero on the submodule corresponding to $H^2_2(\mathfrak{h}'_-, \mathfrak{h}')$, or maps it injectively into $\Lambda^1(\mathfrak{h}'_+) \otimes \mathfrak{h}'$. But by Kostant's version of BBW, the submodule corresponding to $H^2_2(\mathfrak{h}'_-, \mathfrak{h}')$ has multiplicity 1 in $\Lambda^*(\mathfrak{h}'_+) \otimes \mathfrak{h}'$, and hence $\kappa_H \in \ker([\,,] \otimes \mathrm{id})$. Applying Proposition 42 ($\ker([\,,] \otimes \mathrm{id})$ is a Q'-module), we see that the same holds for the full curvature κ, which shows the first statement in the Lemma.

In Chapter 4.1 we noted that the regular normal Cartan connection of a QC^+ structure (integrability is assumed in dimension 7) is automatically torsion-free of type (H', Q'). Hence we have $\kappa(X, Y) \in \mathfrak{q}'$ for all $X, Y \in \mathfrak{h}'_-$. In particular, $\kappa^{(2)}(X, Y) \in \mathfrak{h}'_0$. Then applying Proposition 37 to $\kappa^{(2)}$ for $X, Y \in \mathfrak{h}'_{-1}$ and $Z \in \mathfrak{h}'_{-2}$, the definition of the differential $\partial : C^2(\mathfrak{h}'_-, \mathfrak{h}') \to C^3(\mathfrak{h}'_-, \mathfrak{h}')$ (cf. Definition in Chapter 2.2) gives:

$$\begin{aligned} 0 &= (\partial\kappa^{(2)})(Z, X, Y) \\ &= [Z, \kappa^{(2)}(X, Y)] - [X, \kappa^{(2)}(Z, Y)] + [Y, \kappa^{(2)}(Z, X)] \\ &\quad - \kappa^{(2)}([Z, X], Y) + \kappa^{(2)}([Z, Y], X) - \kappa^{(2)}([X, Y], Z) \\ &= [Z, \kappa^{(2)}(X, Y)]. \end{aligned}$$

Taking an arbitrary $Z \in \mathfrak{h}'_{-2}$ and using the general form for a matrix in $\mathfrak{h}'_0$ which was given in Chapter 4.1, we see that the last line is

$$[Z,\kappa^{(2)}(X,Y)] = [\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}, \begin{pmatrix} a+x & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a+x \end{pmatrix}] \tag{4.36}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2az+zx-xz & 0 & 0 \end{pmatrix} \tag{4.37}$$

Now it follows that (4.37) can only vanish, for arbitrary $z \in Im(\mathbb{H})$, if $a = x = 0$. But matrices of this form in $\mathfrak{h}'_0$ are actually contained in $\mathfrak{p}$, since they annihilate the real light-like vector e_0. Thus $\kappa_H = \kappa^{(2)}$ has values in $\mathfrak{p} \cap \mathfrak{h}'$. Since this is a Q'-module, Proposition 42 again tells us that κ has values in this module, i.e. κ is torsion-free considered as type (G, P). □

Proof of Lemma 83: From the identity (4.17) of Lemma 79, we know that $(\partial^*_{\mathfrak{q}}(\kappa))_2$ vanishes, and moreover, by identity (4.16) of the same Lemma, it suffices to show that the map

$$\psi := pr_{(\mathfrak{h}')^\perp}((\partial^*_{\mathfrak{q}'}\kappa)_1(X))$$

vanishes for an arbitrary $X \in \mathfrak{h}_-$.

Let Ω denote the symplectic form defining $\mathfrak{sp}(2(n''+2), \mathbb{C})$ as a complex subalgebra of $\mathfrak{so}(2(n''+2), \mathbb{C})$. We have the standard identity

$$\mathfrak{sp}(p''+1, q''+1) = \mathfrak{su}(2(p''+1), 2(q''+1)) \cap \mathfrak{sp}(2(n''+2), \mathbb{C}),$$

and using the splitting $2A = (A + \Omega A \Omega) + (A - \Omega A \Omega)$, for A any matrix in $\mathfrak{su}(2(p''+1), 2(q''+1))$, we can identify the subspace $(\mathfrak{h}')^\perp \subset \mathfrak{h}$ as the set of those matrices which anti-commute with multiplication by j (and hence also k) on $\mathbb{H}^{p''+1,q''+1} = \mathbb{C}^{p''+2,q''+2}$.

Moreover, since κ is torsion-free of type (G, P), from definition (2.30) we see that $\psi \in [\mathfrak{q}, \mathfrak{h}_+] \subset \mathfrak{h}_+$. The subalgebra $\mathfrak{h}_+$ can be characterized as those maps in $\mathfrak{h}$ which map all vectors in the complex orthocomplement $L_{\mathbb{C}}^{\perp_{\mathbb{C}}}$ into $L_{\mathbb{C}}$. But the subspace $L_{\mathbb{H}}^{\perp_{\mathbb{H}}}$ is contained in the former subspace, and since ψ anti-commutes with both j and k, the image $\psi(L_{\mathbb{H}}^{\perp_{\mathbb{H}}})$ is a quaternionic subspace, contained in the complex line $L_{\mathbb{C}}$, and must be the 0 subspace.

Therefore, the map ψ is determined on the quotient $\mathbb{H}^{p''+1,q''+1}/L_{\mathbb{H}}^{\perp_{\mathbb{H}}}$. Let $v_0 \in L_{\mathbb{C}}$ be a non-zero vector, and let $x_0 \in \mathbb{H}^{p''+1,q''+1}$ be its dual vector: $< v_0, x_0 >= 1$. Letting $w_0 := \psi(x_0)$, then $\{x_0, jx_0\}$ induce a complex basis of the quotient space, and the map ψ is determined by $(x_0, jx_0) \longmapsto (w_0, -jw_0)$. On the other hand, since $jx_0 \in L_{\mathbb{C}}^{\perp_{\mathbb{C}}}$, we must have $w_0 \in L_{\mathbb{C}}$, i.e. $w_0 = jz_0v_0$ for some $z_0 \in \mathbb{C}$. Therefore, the map ψ in question is determined by $\psi : (x_0, jx_0) \longmapsto (z_0jv_0, z_0v_0)$, which is easily seen to be symmetric with respect to $<,>_{\mathbb{C}}$. Thus, $\psi \in \mathfrak{h}$ only if it is identically zero.

We have thus shown that $(\partial^*_{\mathfrak{q}}\kappa)_1 = (\partial^*_{\mathfrak{q}}\kappa)_2 = 0$. Then the final argument in the proof of Theorem 2.5 of [18] (the preceding argument is just a symplectic version of that argument), shows that $\partial^*_{\mathfrak{p}} \circ \kappa$ vanishes. □

4.4 Weyl structures for Fefferman spaces

In this Section, we give a recipe for relating the (exact) Weyl structures of a parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of general parabolic type (H, Q) to exact Weyl structures on the generalized Fefferman space $(\mathcal{P}, \pi, M, \omega)$ of type (G, P), whenever $H \subseteq G$ acts transitively on G/P and $Q \supseteq P^H$. In the following Section, we will apply this to give a geometrically simple argument showing that the generalized conformal Fefferman space of a CR^+ geometry, as discussed in Chapter 4.1, is a conformal covering of the classical Fefferman space as defined, e.g. in [42].

Let (H, Q) and (G, P) be as above, and let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of H and G, respectively. Suppose $\mathfrak{h}$ has a $|k|$-grading induced by the Lie algebra $\mathfrak{q}$ of Q, and that $\mathfrak{g}$ has a $|\tilde{k}|$-grading induced by the Lie algebra $\mathfrak{p}$ of P. To relate the Weyl structures, we need a certain type of inclusion of the Lie algebras:

Definition 84 *A* graded inclusion *of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the parabolic pairs (H, Q) and (G, P) is an injective Lie algebra homomorphism*

$$\varphi : \mathfrak{h} \hookrightarrow \mathfrak{g}$$

together with a decomposition $\varphi_\bullet = (\varphi_{-\tilde{k}}, \ldots, \varphi_{\tilde{k}})$ such that $\varphi_i(X) \in \mathfrak{g}_i$ for all $-\tilde{k} \leq i \leq \tilde{k}$ and all $X \in \mathfrak{h}$. Equivalently, for the grading element $\tilde{\varepsilon}_0$ defining the grading of $\mathfrak{g}$, $\varphi_\bullet$ satisfies

$$[\tilde{\varepsilon}_0, \varphi_i] = i\varphi_i.$$

To define a map of Weyl structures on $(\mathcal{Q}, \pi', N, \omega')$ to Weyl structures on the Fefferman space, it is most convenient to use the third characterization of a Weyl structure given by (2.56) in Chapter 2.4. So a Weyl structure on $(\mathcal{Q}, \pi', N, \omega')$ is given by a Q-equivariant isomorphism

$$\mathcal{E}_\bullet : \mathcal{Q} \times (\mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_+) \to \mathcal{Q} \times \mathfrak{h}.$$

Then the associated Weyl structure on the Fefferman space is precisely the P-equivariant isomorphism $\tilde{\mathcal{E}_\bullet}$ which makes the following diagram commute:

$$\begin{array}{ccc}
\mathcal{P} \times (\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+) & \xrightarrow{\tilde{\mathcal{E}_\bullet}} & \mathcal{P} \times \mathfrak{g} \\
\Phi_\bullet \uparrow & & \uparrow \Phi \\
\mathcal{Q} \times (\mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_+) & \xrightarrow{\mathcal{E}_\bullet} & \mathcal{Q} \times \mathfrak{h}
\end{array}$$

where $\Phi_\bullet := \iota \times \varphi_\bullet$ and $\Phi := \iota \times \varphi$ are induced by the usual inclusion

$$\iota : \mathcal{Q} = \mathcal{P}^H \hookrightarrow \mathcal{P}.$$

Explicitly, $\tilde{\mathcal{E}_\bullet}$ is given as follows. First, by Q-invariance we may assume that the first factor of $\mathcal{E}_\bullet$, composed with projection onto the first component, which maps

$$\mathcal{Q} \to \mathcal{Q},$$

is the identity, and the same will hold for $\tilde{\mathcal{E}_\bullet}$. We can thus abuse notation and write

$$\mathcal{E}_\bullet^{-1}(u) : \mathfrak{h} \to (\mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h})$$

to denote the linear isomorphism, depending smoothly on $u \in \mathcal{Q}$, which is given by composing $\mathcal{E}_\bullet^{-1}$ with projection onto the second component.

Using this, we define a linear isomorphism

$$\tilde{\mathcal{E}_\bullet}^{-1}(u) : \mathfrak{g} \to (\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+)$$

for $u \in \mathcal{Q} \subseteq \mathcal{P}$ as follows: For $X \in \mathfrak{h} \subset \mathfrak{g}$, let

$$\tilde{\mathcal{E}_\bullet}^{-1}(u)(X) := (\varphi_{-\tilde{k}}(\mathcal{E}_\bullet^{-1}(u)(X)), \dots, \varphi_{\tilde{k}}(\mathcal{E}_\bullet^{-1}(u)(X))).$$

By transitivity of H on G/P, we see in particular that

$$q \circ \varphi : \mathfrak{h} \to (\mathfrak{g}/\mathfrak{p}) \cong \mathfrak{g}_-$$

is onto. From the isomorphisms $\mathfrak{h}_+ \cong (\mathfrak{h}_-)^*$ and $\mathfrak{g}_+ \cong (\mathfrak{g}_-)^*$, moreover, this implies that the vector subspace $\mathfrak{g}_- \oplus \mathfrak{g}_+ \subseteq \varphi_\bullet(\mathfrak{h})$. Thus, from bijectiveness and linearity, we have a well-defined map

$$\tilde{\mathcal{E}_\bullet}(u)_{|\mathfrak{g}_- \oplus \mathfrak{g}_+} : \mathfrak{g}_- \oplus \mathfrak{g}_+ \to \mathfrak{g}.$$

Now, $\tilde{\mathcal{E}_\bullet}(u)(\mathfrak{g}_- \oplus \mathfrak{g}_+)$ is a non-degenerate subspace of $\mathfrak{g}$ with respect to the Killing form, and we define, for any $X = \tilde{\mathcal{E}_\bullet}(u)(X_- + X_+) + X^\perp \in \mathfrak{g}$ split according to this decomposition,

$$\tilde{\mathcal{E}_\bullet}^{-1}(u)(X) = X_- + X^\perp + X_+$$

where we take $X^\perp \in \mathfrak{g}_0$.

The linear isomorphisms $\tilde{\mathcal{E}_\bullet}(u)$ thus defined for $u \in \mathcal{P}^H$, are Q-equivariant since $\mathcal{E}_\bullet$ has this property, and they clearly make the diagram commute. Extending by P-equivariance to all points of $\mathcal{P}$ gives the desired Weyl structure $\tilde{\mathcal{E}_\bullet}$.

To make use of these naturally induced Weyl structures to describe explicitly the conformal geometry of the Fefferman spaces we're interested in, we need a result on the exactness of $\tilde{\mathcal{E}_\bullet}$. Recall from Chapter 2.4 that a Weyl structure given in the form $\mathcal{E}_\bullet$ as it is here, gives a Q-invariant lift of the splitting of the Cartan connection:

$$\begin{aligned} \mathcal{E}_\bullet^{-1} \circ \omega' &= \omega'_{\mathfrak{h}_-} + \omega'_{\mathfrak{h}_0} + \omega'_{\mathfrak{h}_+} \\ &= (\pi')^*\theta^\sigma + (\pi'_+)^*\omega^\sigma + (\pi')^*\mathrm{P}^\sigma, \end{aligned}$$

where $\sigma : \mathcal{Q}_0 \to \mathcal{Q}$ is the corresponding Weyl structure, in the form of a Q_0 equivariant section of the Q_+-PFB $\pi'_+ : \mathcal{Q} \to \mathcal{Q}_0$. Moreover, given a representation $\lambda : Q_0 \to \mathbb{R}^+$ inducing the bundle of scales $\mathcal{L}^\lambda$, there is a (trivial) extension $\bar{\lambda} : Q \to \mathbb{R}^+$ given by $\bar{\lambda} = \lambda \circ q$, for $q : Q \to Q/Q_+ \cong Q_0$ the quotient map.

Then given a scaling element $\varepsilon_\lambda \in \mathfrak{h}_0$ corresponding to a representation $\lambda : H_0 \to \mathbb{R}^+$ and a bundle of scales $\mathcal{L}^\lambda$ on N, the definition of an exact Weyl structure can clearly be reformulated as:

Proposition 85 *A Weyl structure $\mathcal{E}_\bullet$ is exact if and only if the connection on $\mathcal{L}^\lambda \cong \mathcal{Q}/\mathrm{Ker}(\bar{\lambda})$ to which $\lambda' \circ \omega'_{\mathfrak{h}_0} \in \Omega^1(\mathcal{Q})$ descends, has trivial holonomy.*

We can now formulate the main result of this Section, giving induced ***exact*** Weyl structures on the Fefferman space. Note, for the formulation of the conditions, that we assume fixed algebraic Weyl structures for both (H, Q) and (G, P) (giving the gradings on the Lie algebras). In particular, this allows us to identify $Q_0 = Q/Q_+$ and $P_0 = P/P_+$ as subspaces of Q and P, respectively.

Proposition 86 *Let $(\mathcal{P}, \pi, M, \omega)$ be a Fefferman space of parabolic type (G, P), induced by a parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of type (H, Q), with a graded inclusion $\varphi_\bullet$ of $\mathfrak{h}$ in $\mathfrak{g}$ as above. Suppose that scaling elements $\varepsilon_\lambda \in \mathfrak{z}(\mathfrak{h}_0)$ and $\varepsilon_{\tilde{\lambda}} \in \mathfrak{z}(\mathfrak{g}_0)$ exist for the base geometry and the Fefferman space, respectively, such that $\varepsilon_\lambda \in \mathfrak{z}(\mathfrak{h}_0) \cap \mathfrak{p}$ and such that*

$$\begin{array}{ccc} Q_0 \cap P & \xrightarrow{\quad \tilde{q} \circ \iota \quad} & P_0 \\ & \lambda \searrow \qquad \swarrow \tilde{\lambda} & \\ & \mathbb{R}^+ & \end{array}$$

commutes. Then the induced Weyl structure $\tilde{\mathcal{E}_\bullet}$ on $(\mathcal{P}, \pi, M, \omega)$ is exact whenever $\mathcal{E}_\bullet$ is an exact Weyl structure on $(\mathcal{Q}, \pi', N, \omega')$.

Proof: Consider the projection

$$p : M \to N$$

given naturally by the Fefferman construction. Clearly it suffices to prove that under the above assumptions, the scale bundle $\mathcal{L}^{\tilde{\lambda}}$ defined on M by $\tilde{\lambda}$ is a pull-back of the scale bundle $\mathcal{L}^{\lambda}$ with respect to the map p, and that the connection on $\mathcal{L}^{\tilde{\lambda}}$ to which $\tilde{\lambda}' \circ \omega_{\mathfrak{g}_0} \in \Omega^1(\mathcal{P})$ descends, is induced by this pullback.

By differentiating the commutativity relation in the assumptions of the Proposition, we get:

$$\lambda' = \tilde{\lambda}' \circ \varphi_0.$$

Combining this with the commutativity condition which defined $\tilde{\mathcal{E}_\bullet}$, we see that the one-form $\lambda' \circ \omega'_{\mathfrak{h}_0} \in \Omega^1(\mathcal{Q})$, which descends to the principal bundle connection on $\mathcal{L}^{\lambda}$, is the pull-back of the one-form $\tilde{\lambda}' \circ \omega_{\mathfrak{g}_0}$, which induces the principal bundle connection on $\mathcal{L}^{\tilde{\lambda}}$. Hence if we can show that $\mathcal{L}^{\tilde{\lambda}}$ fibers over $\mathcal{L}^{\lambda}$, covering the fiber bundle projection $p : M \to N$, then the result follows.

The commutativity assumption in the Proposition implies that

$$\mathrm{Ker}(\bar{\lambda}) \cap P = H \cap \mathrm{Ker}(\bar{\tilde{\lambda}}).$$

and hence that

$$\mathcal{Q}/(\mathrm{Ker}(\bar{\lambda}) \cap P) \cong \mathcal{P}/\mathrm{Ker}(\bar{\tilde{\lambda}}) \cong \mathcal{L}^{\tilde{\lambda}}.$$

On the other hand,

$$\mathcal{L}^{\lambda} \cong \mathcal{Q}/\mathrm{Ker}(\bar{\lambda}),$$

which induces a fibration $\mathcal{L}^{\tilde{\lambda}} \to \mathcal{L}^{\lambda}$ with fiber diffeomorphic to $\mathrm{Ker}(\bar{\lambda})/(\mathrm{Ker}(\bar{\lambda}) \cap P)$. But from the assumption $\varepsilon_\lambda \in \mathfrak{p}$, it follows that this fiber is isomorphic to $Q/(P \cap H)$, from which it's clear that $\mathcal{L}^{\tilde{\lambda}}$ is a pull-back of $\mathcal{L}^{\lambda}$ by the projection p. □

4.5 Relation to the classical Fefferman metric

To conclude, we apply the techniques of the previous Section to explicitly identify the conformal structure of the Fefferman space induced by the canonical parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ associated to a CR^+ manifold $(N, \mathcal{V}, J)$. To relate this to the conformal class of metrics given directly by the classical Fefferman construction, first we note that our Fefferman space M is a $(n'+2)$-fold covering of the classical Fefferman space F.

This can be seen by considering a pseudo-Hermitian form $\theta \in \Omega^1(N)$ for the underlying CR structure $(N, \mathcal{V}, J)$. This fixes a reduction of the filtered CR frame bundle to a $U(p', q')$-PFB, which we'll denote as $\mathcal{U}^\theta(N, \mathcal{V}, J)$. The filtered CR^+ frame bundle $\mathcal{Q}_0$ is by definition a $(n'+2)$-fold covering of the filtered CR frame bundle, and thus lifting the reduction given by θ also gives a reduction of this bundle to a principal $Q_0/\mathbb{R}^+$ fiber bundle which is a $(n'+2)$-fold covering of $\mathcal{U}^\theta(N, \mathcal{V}, J)$.

Topologically, the classical Fefferman space F is the $U(1)$-PFB $f : F \to N$ given by the real projectivization of the $\mathbb{C}^*$ bundle $\mathcal{K} - \{0\}$, for $\mathcal{K}$ the canonical bundle defined in Chapter 4.1. Given a pseudo-Hermitian form θ for $(N, \mathcal{V}, J)$, we have the Levi form L_θ of the pseudo-Hermitian structure, which is non-degenerate of signature (p', q'), and using this we can define a unitary frame to be a local complex basis of $T^{1,0}$,

$$\{E_1, \ldots, E_{n'}\},$$

with respect to which $L_\theta^{\mathbb{C}}$ has the form $I_{p',q'}$, and such frames define the $U(p', q')$-PFB $\mathcal{U}^\theta(N, \mathcal{V}, J)$. If we let $\{\theta^1, \ldots, \theta^{n'}\}$ be the dual one-forms in $T^*N \otimes \mathbb{C}$ to such a local basis, then the form $\tau = \theta^{\mathbb{C}} \wedge \theta^1 \wedge \ldots \wedge \theta^{n'}$ defines a local section of the canonical bundle $\mathcal{K}$.

From this, we see that F can equivalently be defined as the associated $U(1)$ bundle given by the determinant representation:

$$F = \mathcal{U}^\theta(N, \mathcal{V}, J) \times_{(U(p',q'),\mathrm{det})} S^1.$$

From the descriptions above, it can also be seen that the manifold from the generalized Fefferman space, $M = \mathcal{Q}/P^H$, can be associated to the principal $\mathcal{Q}_0/R^+$ via the determinant representation applied to the unitary matrix in the "middle" of Q_0. This identifies M as the $(n'+2)$nd root of the $U(1)$-bundle $F \to N$:

$$M = \sqrt[(n'+2)]{F}$$

and we denote the natural $(n'+2)$-fold covering of $U(1)$-bundles by

$$\digamma : M \to F.$$

Now, the Webster connection form $\omega^\theta \in \Omega^1(\mathcal{U}^\theta(N, \mathcal{V}, J), \mathfrak{u}(p', q')$ for a choice of pseudo-Hermitian form θ, induces a principal bundle connection form on F, which we denote by A^θ, and given a local unitary frame s and its induced local section τ_s of the canonical bundle as above, we denote the induced local section of F by τ_s', and we have:

$$(\tau_s')^* A^\theta = -\mathrm{tr}(\omega^\theta(s)).$$

The $U(1)$ principal bundle connections on F form an affine space modeled on the space of one-forms on N with purely imaginary values. The Fefferman connection form on F, A^F, is defined, given a choice of pseudo-Hermitian form θ, by:

$$A^F := A^\theta - \frac{iR^\theta}{2(n'+1)}\theta,$$

where R^θ denotes the scalar curvature of the Webster connection ω^θ. Then the *Fefferman metric* on F associated to the pseudo-Hermitian form, is given by

$$f_\theta := f^* L_\theta - i\frac{4}{n'+2} f^*\theta \circ A^F. \qquad (4.38)$$

Here, $\circ$ denotes the symmetric tensor product. The conformal class of metrics $[f_\theta]$ thus defined on the classical Fefferman space F is invariant under choice of (positively oriented) pseudo-Hermitian form θ. More precisely, cf. Theorem 5.17 of [42], we have $f_{e^{2\phi}\theta} = e^{2\phi} f_\theta$ for any smooth function $\phi \in C^\infty(N)$. Now we relate the classical Fefferman construction to our conformal Fefferman space:

Proposition 87 *Let $(N, \mathcal{V}, J)$ be a CR manifold which also has a CR^+ structure, with associated canonical parabolic geometry $(\mathcal{Q}, \pi', N, \omega')$ of type (H, Q). Let $(\mathcal{P}, \pi, M, \omega)$ be the (parabolic) conformal Fefferman space induced by this parabolic geometry, and let (F, c) be the classical Fefferman space associated to the CR manifold. Then $\digamma : M \to F$ is a conformal covering.*

More precisely, let θ be a pseudo-Hermitian form for the CR structure, inducing an (exact) Weyl structure $\mathcal{E}^\theta_\bullet$ on $(\mathcal{Q}, \pi', N, \omega')$ and a pseudo-Riemannian metric $f_\theta \in c$. Then $\digamma : M \to F$ gives an isometric covering of (F, f_θ) by the pseudo-Riemannian metric on M defined by the induced Weyl structure $\tilde{\mathcal{E}}^\theta_\bullet$.

Proof: Although the assumptions of Proposition 86 at first glance seem rather elaborate and technical, in fact in the CR case they are all quite naturally satisfied. A graded inclusion $\varphi_\bullet : \mathfrak{h} \to \mathfrak{g}$ was defined in Chapter 4.2. We use the simplest scaling elements, namely the grading elements:

$$\varepsilon_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \varepsilon_{\tilde{\lambda}} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is straightforward to see that ε_λ and $\varepsilon_{\tilde{\lambda}}$ thus defined, satisfy the assumptions of Proposition 86. Hence the induced Weyl structure $\tilde{\mathcal{E}}^\theta_\bullet$ on $(\mathcal{P}, \pi, M, \omega)$, coming from a pseudo-Hermitian form θ on the CR manifold $(N, \mathcal{V}, J)$, is exact.

In particular, $\tilde{\mathcal{E}}^\theta_\bullet$ defines a pseudo-Riemannian metric (which we'll denote by γ_θ) contained in the conformal class on M induced by the parabolic geometry $(\mathcal{P}, \pi, M, \omega)$. Consider the splitting of the Cartan connection ω:

$$(\tilde{\mathcal{E}}^\theta_\bullet)^{-1} \circ \omega = \omega_{\mathfrak{g}_-} + \omega_{\mathfrak{g}_0} + \omega_{\mathfrak{g}_+}$$

given by the Weyl structure $\tilde{\mathcal{E}}^\theta_\bullet$. We can describe the metric γ_θ on M by using the first component of this splitting, which is the pull-back by π of the soldering form on the orthogonal frame bundle of γ_θ. Explicitly, for $X, Y \in TM$, and choosing $\tilde{X}, \tilde{Y} \in T\mathcal{P}$ such that $\pi_* \tilde{X} = X, \pi_* \tilde{Y} = Y$, we have

$$(\gamma_\theta)_{\pi(u)}(X, Y) = \omega_{\mathfrak{g}_-}(u)(\tilde{X}) \cdot (\omega_{\mathfrak{g}_-}(u)(\tilde{Y}))^{\psi t}$$

where we identify an element $A \in \mathfrak{g}_-$ with a column vector via the isomorphism $\mathfrak{g}_- \cong \mathbb{R}^{p+q+2}$.

Using this realization of the metric γ_θ, we can describe it in terms of geometric objects determined by the pseudo-Hermitian structure $(N, \mathcal{V}, J, \theta)$. Let $\xi \in \Gamma(TN)$ be the Reeb vector field determined by θ and let $E_1, \ldots, E_{n'}, iE_1, \ldots, iE_{n'}$ be a local unitary basis of $\mathcal{V}$ with respect to the metric g_θ. To use the splitting of ω above to give γ_θ in terms of these vectors, there is a preferred lift of these local vector fields to the bundle $\mathcal{Q}$ which are natural. Namely, we consider the splitting of the Cartan connection ω' given by $\mathcal{E}^\theta_\bullet$:

$$(\mathcal{E}^\theta_\bullet)^{-1} \circ \omega' = \omega'_{\mathfrak{h}_-} + \omega'_{\mathfrak{h}_0} + \omega'_{\mathfrak{h}_+}. \tag{4.39}$$

In particular, the "non-negative" part,

$$\omega'_{\mathfrak{q}} = \omega'_{\mathfrak{h}_0} + \omega'_{\mathfrak{h}_+} \in \Omega^1(\mathcal{Q}, \mathfrak{q}),$$

is seen to give a principal bundle connection on $\mathcal{Q}$. Letting

$$\tilde{\xi}, \tilde{E}_1, \ldots, \tilde{E}_{n'}, i\tilde{E}_1, \ldots, i\tilde{E}_{n'}$$

be the horizontal lifts of the above vector fields on N with respect to this connection, then by definition, $(\mathcal{E}^\theta_\bullet)^{-1} \circ \omega'(\zeta) = \omega'_{\mathfrak{h}_-}(\zeta)$ for any of these vectors ζ. Using the fact that $\omega'_{\mathfrak{h}_-}$ is the pull-back of the soldering form on N given by the pseudo-Hermitian form θ, moreover, we get the identities:

$$(\mathcal{E}^\theta_\bullet)^{-1} \circ \omega'(\tilde{\xi}) = X_0 := \mathfrak{h}_{-2}(i); \tag{4.40}$$

$$(\mathcal{E}^\theta_\bullet)^{-1} \circ \omega'(\tilde{E}_j) = X_j := \mathfrak{h}_{-1}(e_j); \tag{4.41}$$

$$(\mathcal{E}^\theta_\bullet)^{-1} \circ \omega'(i\tilde{E}_j) = X_{n'+j} := \mathfrak{h}_{-1}(ie_j); \tag{4.42}$$

where $X_0, \ldots, X_{2n'}$ are the standard basis elements of $\mathfrak{h}_-$ as defined in Chapter 4.2.

Now, from the form of φ_{-1}, and the defining property of the induced Weyl structure $\tilde{\mathcal{E}}^\theta_\bullet$, we see immediately that the form $\omega_{\mathfrak{g}_-}$ given by this Weyl structure, maps the vectors $\tilde{E}_j, i\tilde{E}_j$ to a set of pseudo-orthonormal basis vectors in $\mathfrak{g}_- \cong \mathbb{R}^{p,q}$, while it maps the vector $\tilde{\xi}$ to a light-like vector whose dual is given by the vector $\tilde{X}_{2n'+1}$ defined in Chapter 4.2. In particular, if we fix an element

$$\delta = \begin{pmatrix} i & 0 & 0 \\ 0 & iB_0 & 0 \\ 0 & 0 & i \end{pmatrix} \in \mathfrak{h}_0,$$

then $\varphi_{-1}(\delta) = \tilde{X}_{2n'+1}$; let $\tilde{\delta}$ be its fundamental vector field on $\mathcal{Q}$, then we get the following identities for γ_θ, where $\pi : \mathcal{P} \to M$ is the usual projection map and $\mathcal{Q} = \mathcal{P}^H$ is considered as a submanifold of $\mathcal{P}$:

$$\gamma_\theta(\pi_*(\tilde{E}_j), \pi_*(\tilde{E}_k)) = \gamma_\theta(\pi_*(i\tilde{E}_j), \pi_*(i\tilde{E}_k)) = \delta_{jk}; \tag{4.43}$$

$$\gamma_\theta(\pi_*(\tilde{\xi}), \pi_*(\tilde{\delta})) = 1; \tag{4.44}$$

for all $1 \leq j, k \leq n'$, and where all other pairings not listed give zero (in particular, $\pi_*(\tilde{\xi})$ and $\pi_*(\tilde{\delta})$ are light-like with respect to γ_θ.

To relate the above-described metric γ_θ to the classical Fefferman metric f_θ via the covering $\digamma : M \to F$, we need explicit information about the horizontal lifts used above, which in turn comes from information about the explicit form of the principal bundle connection $\omega'_{\mathfrak{q}} = \omega'_{\mathfrak{h}_0} + \omega'_{\mathfrak{h}_+}$ defining the lifts.

In [36] as well as [34], formulae are given which, analogous to the matrix form given in (3.9) in Chapter 3.1 for conformal geometry, express the standard Tractor connection of a CR^+ structure in terms of an induced decomposition and geometric quantities of a pseudo-Hermitian form θ for the underlying CR structure. We will only need the component corresponding to $\omega'_{\mathfrak{h}_0}$, which corresponds to the "diagonal" component of the affine connection

$$\nabla^{\mathfrak{h}} : \mathcal{T}^{CR}(N) \to T^*N \otimes \mathcal{T}^{CR}(N),$$

where $\mathcal{T}^{CR}(N)$ is the CR Tractor bundle, which is decomposed, via the pseudo-Hermitian form θ, as

$$\mathcal{T}^{CR}(N) \cong \sqrt[-(n'+2)]{\mathcal{K}} \oplus TN \oplus \sqrt[(n'+2)]{\mathcal{K}}$$

for $\mathcal{K}$ the canonical bundle of $(N, \mathcal{V}, J)$ introduced in Chapter 4.1, which we assume has a $(n'+)$nd root. From the formulae (5.11)-(5.13) in [36], the diagonal component can be written as:

$$\nabla^{\mathfrak{h}_0}_X = \begin{pmatrix} \nabla^\theta_X & 0 & 0 \\ 0 & \nabla^\theta_X & 0 \\ 0 & 0 & \nabla^\theta_X \end{pmatrix} + i\theta(X) \begin{pmatrix} \frac{R^\theta}{2(n'+1)(n'+2)} & 0 & 0 \\ 0 & \frac{R^\theta}{2(n'+1)(n'+2)} - \mathrm{P}^\theta & 0 \\ 0 & 0 & \frac{R^\theta}{2(n'+1)(n'+2)} \end{pmatrix},$$

where P^θ is the *CR Schouten tensor*, defined analogously to the conformal Schouten tensor by:

$$\mathrm{P}^\theta := \frac{1}{n'+2}(Ric^\theta - \frac{R^\theta}{2(n'+1)} g_\theta),$$

and satisfying: $\mathrm{tr}\mathrm{P}^\theta = \frac{1}{2(n'+1)} R^\theta$. Here ∇^θ denotes the affine connection induced by the Webster connection form ω^θ associated to the pseudo-Hermitian form θ. The affine connection acts naturally on $\mathcal{K}$ and its roots, which is what's denoted in the upper left and lower right entries.

Using the above formula and remarks, we can convince ourselves that $\omega'_{\mathfrak{h}_0} \in \Omega^1(\mathcal{Q}, \mathfrak{h}_0)$ is the pull-back (by the $(n'+2)$-fold covering map composed with the projection maps $\mathcal{Q} \to \mathcal{Q}_0 \to \mathcal{Q}_0/\mathbb{R}^+$) of the principal connection form

$$\omega^F := \omega^W - i(\mathrm{P}^\theta \otimes \theta),$$

and one can also check that ω^F induces, via the determinant representation, the Fefferman connection form $A^F \in \Omega(F; \mathfrak{u}(1))$.

Therefore, taking the horizontal lift to M of a vector field on N with respect to the pull-back of this connection form, $\digamma^* A^F$, is equivalent to first taking the horizontal lift to $\mathcal{Q}$ with respect to $\omega'_{\mathfrak{q}}$, and then projecting with π_* to a vector field on M. (Note that the $\mathfrak{h}_+$-component of $\omega'_{\mathfrak{q}}$ doesn't effect anything, since φ_{-1} restricted to $\mathfrak{h}_+$ is identically zero.) On the other hand, from the condition $b = -\frac{1}{2}\mathrm{tr}_{\mathbb{C}}(A_0)$ for a matrix in $\mathfrak{h}_0$ of the form

$$\begin{pmatrix} a+ib & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & -a+ib \end{pmatrix},$$

it follows that the fundamental vector field $\tilde{\delta}$ projects onto $(1/2)\tilde{i}$, the fundamental vector field on M of $(i/2) \in \mathfrak{u}(1)$. And since $\digamma$ is a $(n'+2)$-fold covering, this vector projects onto $\frac{n'+2}{2}\tilde{i}^F$, the fundamental vector field on F. Then we can translate the formulae in (4.43) and (4.44) for γ_θ to get the following identities, where ξ is the Reeb vector field:

$$\gamma_\theta(E_j^*, E_k^*) = \gamma_\theta(iE_j^*, iE_k^*) = \delta_{jk};$$
$$\gamma_\theta(\xi^*, \tilde{i}) = 2.$$

Using the formula (4.38) for f_θ, the above information about projection of fundamental vector fields under $\digamma$ and the fact that this covering preserves horizontal lifts, we get in comparison:

$$f_\theta(F_*(E_j^*), F_*(E_j^*)) = f_\theta(F_*(iE_j^*), F_*(iE_k^*)) = \delta_{jk};$$
$$f_\theta(F_*(\xi^*), F_*(\tilde{i})) = 2.$$

□

Chapter 5

Conclusions and perspectives

We have given a Fefferman construction principle for conformal manifolds with (irreducible) symplectic holonomy, and at the same time shown that conformal manifolds with this holonomy are always locally isomorphic to the Fefferman construction. Besides extending the conformal holonomy correspondence for Fefferman constructions starting with a CR manifold, our presentation has the advantage of putting the construction in the context of a possible classification for all (connected) irreducible conformal holonomy groups acting transitively on the Möbius sphere. In this concluding Chapter, we discuss the possible avenues of further work to be done on this and other aspects of conformal holonomy theory.

An obvious task is to investigate the other cases indicated by the list obtained in Chapter 3.4. For each of these groups H, and any parabolic subgroup $Q \supseteq P^H$, of course a parabolic geometry of type (H, Q) gives rise to a conformal Fefferman space of the appropriate signature. With explicit representations of the Lie algebras, one should try to establish what kind of geometric structures correspond to the parabolic pair; whether a normal, torsion-free Cartan connection of type (H, Q) induces a normal Cartan connection of conformal type (and the vice-versa); etc.

This work has already been done for two other groups on the list: $G_{2,2} \subset SO_0(3,4)$ and $Spin(3,4) \subset SO_0(4,4)$. In [50], P. Nurowski constructed a conformal class of metrics of signature $(2,3)$ from an undetermined system of ODEs of a certain type. Nurowski's construction took off from Cartan's "five variables paper" [25], which in modern terminology associated to every generic rank 2 distribution on a 5-manifold N, a Cartan geometry of type $(G_{2,2}, Q)$ for a certain parabolic subgroup Q, cf. [20]. In this case, $Q = P \cap G_{2,2}$, and the conformal Fefferman structure induced by this geometry is defined on the same manifold $M = N$. The calculations of [50] show that the induced conformal Cartan connection is normal.

In [9], R. Bryant constructed a conformal a conformal class of metrics determined by generic rank 3 distributions on 6-manifolds. Again, this is a problem inspired by work of Cartan; such structures have associated to them a canonical Cartan geometry of type $(Spin(3,4), Q)$, where Q is the parabolic subgroup which stabilizes a null line in $\mathbb{R}^{4,4}$, considering $Spin(3,4)$ as an irreducible subgroup of $SO_0(4,4)$. For this reason again, the conformal Fefferman space is the same as the original 6-manifold. Bryant's calculations show that the induced conformal Cartan connection is normal. Moreover, it is proved that a normal conformal Cartan connection for a conformal structure of signature $(3,3)$, if it admits a Cartan reduction to $Spin(3,4)$, gives a normal Cartan connection of this type. I.e., the conformal holonomy correspondence is complete for this group.

Beyond the basic question of whether the conformal holonomy correspondence holds for certain irreducible, transitive subgroups of $SO_0(p+1, q+1)$, one would also hope to gain a better understanding of the geometries involved – that of the base parabolic geometry, as well as the explicit form of the conformal Fefferman metric. For the second part, the results on Weyl structures given here should prove useful, as well as methods from Tractor calculus as in, e.g. [18].

In the case of QC structures, though, we lack an explicit description of the canonical Cartan connection, which stands in the way of solving a number of natural and interesting questions. For example, note that the calculations in Chapter 4.3 also establish that the Fefferman space of type (H, Q) over a QC manifold (which has fibers isomorphic to S^2) is induced by a Cartan connection which is normal and torsion-free, and hence determines an integrable CR structure of the appropriate signature. On the other hand, a central part of Biquard's results on QC structures was the construction of such a bundle with CR structure (cf. Chapter II.5 of [6]: "CR twistors").

One expects that the CR structure of Biquard's twistor space over a QC manifold coincides with the induced CR structure via parabolic geometries, and the general recipe of Chapter 4.4 should be applicable in verifying this, but some information about the canonical QC Cartan connection (its explicit form with respect to a "QC scale") is needed for this, which is evident from the role played by the Biquard connection in constructing the CR twistors. Similarly, such information is required to relate the conformal Fefferman space of a QC manifold described here, to the explicit class of metrics defined in Chapter II.6 of [6]. It should be possible to give the canonical Cartan connection in terms of exact Weyl structures for QC (or QC^+) structures, using bundles of scales and the Biquard connection (cf. [6], [37]), in analogy with the calculations for CR structures.

Information about the other parabolic geometries is relevant in particular for the problem in conformal holonomy of presenting examples of conformal manifolds (or better yet, compact ones) which realize the ***full*** holonomy groups for those in the list above. To our knowledge, non-compact examples are not even worked out for the holonomy group $SU(p'+1, q'+1)$, which evidently can be done by constructing a CR manifold of generic type, i.e. whose canonical Cartan connection has the full holonomy.

Finally, there is the question of irreducible conformal holonomy groups which don't act transitively on the Möbius sphere. At one extreme, the question is if such groups can occur at all. Alternatively, one would hope to formulate and prove results about the geometric structure of spaces having such holonomy. For example, in Riemannian signature there is a geometric proof (cf. [51]) of the fact, which follows from the Berger list, that any Riemannian manifold with irreducible holonomy which doesn't act transitively on the sphere, is a local symmetric space. It would be very nice (but probably over-ambitious) to find an analogous result for conformal manifolds which could be generalized to all signatures.

A more realistic approach to the last questions is to try to answer them in particular signatures, e.g. Lorentzian signature. In particular, it can be shown that $SU(1, n+1)$ has no irreducible connected proper subgroups. Thus both the problem of showing that a strictly pseudo-convex CR manifold has generic holonomy, and that of describing the possible conformal holonomy groups in this signature, are somehow simplified.

An avenue of research which goes in a different direction from basic classification and existence, but which is important especially from the differential geometric viewpoint, is to find global analogs to the mainly local results dealt with here. For one, this entails clarifying the topological conditions giving sharp obstructions to the possibility of globally passing from CR or QC structures to CR^+ and QC^+ structures, respectively. For the QC case, it seems likely that the corresponding results for quaternionic Kähler manifolds – cf. Section 3 of [59] as well as [47], [55] – can be adapted. The other part of studying the global geometry would be looking for (conformally invariant) conditions guaranteeing global extension of the local isomorphisms established in Proposition 70.

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 18. April 2008
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