Conformal Field Theory and Modular Functor

Kenji Ueno

October 12, 2007

Conformal field theory (CFT) is a functor form the category of pointed Riemann surfaces with coordinates to the category of finite dimensional complex vector space, which satisfies several axioms.

Modular functor (MF) is a functor from the category of pointed oriented surfaces with tangent vectors to the category of finite dimensional complex vector space, which satisfies similar properties to those of conformal field theory. Big difference is that CFT does depend on a complex structure but MF only depends on a differentiable structure of a surface. Nevertheless we can construct a modular functor form non-abelian and abelian conformal filed theories.

Let us consider non-abelian conformal field thoery (WSWN model). Fix a complex simple Lie algebra $\mathfrak{g}$ and a positive integer ℓ. Then there exists a finite number of integrable highest weight irreducible representations of the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c$ of level ℓ, where $\mathbb{C}((\xi))$ is the filed of formal Laurent series and c belongs to the center of $\widehat{\mathfrak{g}}$ which acts as $\ell \cdot id$ for representation of level ℓ. Let P_ℓ be the set of highest weights of all the integrable highest weight irreducible representations of $\widehat{\mathfrak{g}}$. Let $M^{(1)}_{g,n}$ be a moduli space of n-pointed compact Riemann surfaces with first order neighbourhoods. Choose $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in P^n_\ell$. Then the conformal field theory determines conformal block bundle $\mathcal{V}^\dagger_{\vec{\lambda}}$ over $M^{(1)}_{g,n}$, which carries a projectively flat connection ∇. If the connection ∇ were flat then the vector space of the flat sections of the vector bundle $\mathcal{V}^\dagger_{\vec{\lambda}}$ over $M^{(1)}_{g,n}$ gives essentially the modular functor. But we have only projectively flat connection. To overcome the difficulty we can use abelian conformal field theory. Abelian conformal field theory is CFT which associate one-dimensional vector space to each n-pointed Riemann surface with coordinates. Again there exists a line bundle $\mathcal{V}^\dagger_{ab}$ over the moduli space $M^{(1)}_{g,n}$ which again carries a projectively flat connection. We can show that both connection forms differ only up to constant multiplication. Hence we can construct fractional power $(\mathcal{V}^\dagger_{ab})^c$ in such a way that $\mathcal{V}^\dagger_{\vec{\lambda}} \otimes_{\mathbb{C}} (\mathcal{V}^\dagger_{ab})^c$ carries a flat connection. Then the vector space of flat sections of this vector bundle gives the desired modular functor.

Now it is known that the modular functor defines Topological Quantum Field Theory (TQFT), which gives invariants of three dimensional manifolds. In case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, TQFT thus obtained is equivalent to Reshetikhin-Turaev TQFT.