Inhaltsverzeichnis

1 [A4] Quasilinear Wave Equations, Membranes, and Supermembranes 2005-2008

1.1 Summary

Models from String Theory and other unified theories are closely interrelated with geometric partial differential equations of elliptic, parabolic and hyperbolic type. This opens up many new unforeseen and intriguing possibilities for exploiting the link between the mathematics of Ricci and Mean Curvature Flows on the one hand, and the physics of Einstein's equations and relativistic (super)membranes on the other.

An "m-brane" describes the motion of an m-dimensional spacelike submanifold in an (m+k+1)-dimensional Lorentzian manifold according to an action integral which is proportional to the (m+1)-dimensional invariant volume of the worldsheet swept out by the evolution of the submanifold (the so-called "Nambu-Goto-action").

These extended relativistic objects are of strong mathematical interest while at the same time they play an important role as "D-branes" in modern string theory, allowing for a better understanding of the nonperturbative aspects of the theory. For the relativistic membrane (m=2) in a target space of m+k+1=11 dimensions there exists a maximally supersymmetric generalisation of this theory, the "supermembrane". This theory extends string theory significantly in that it contains the known superstring models in 10 dimensions (IIA, IIB and heterotic) as well as the maximal supergravity theory in 11 dimensions as special limits. At the same time, it was shown to be

related to (supersymmetric) matrix models. For this reason, it is hoped that supermembrane theory may serve as a basis for a truly non-perturbative and unified description of the fundamental interactions of physics.

From the point of view of differential geometry and analysis the equation of motion for the (bosonic) membrane is the relativistic minimal surface equation, that is the equation of vanishing mean curvature for the world volume. This is a system of quasilinear hyperbolic equations of second order with a structure that is in many respects analogous to the classical Einstein vacuum equations in General Relativity, but has not yet been carefully studied from this point of view. While physicists have been aware of the links between membrane theory and matrix theory for some time, this connection remains to be exploited in the context of pure mathematics and differential geometry.

The first part of this project wants to utilize recent progress in analysis to understand the analytical and geometrical properties of solutions to specific models of theoretical physics such as p-branes in Lorentzian manifolds. This includes the solution of the Cauchy problem for timelike minimal surfaces in a general setting, and also the clarification of the role of diffeomorphisms as gauge transformations. Further open questions concern the longtime behavior of solutions and their asymptotic decay, the stability of stationary solutions and the existence of periodic solutions. In all these questions it will be of particular interest how the nonlinear nature of the equations influences the behavior of solutions.

In a second, and closely related part of the project, the connections between membrane theories and the SU(N) matrix model approximation are to be studied (the membrane is obtained as a limit of the matrix model as N tends to infinity). Among the outstanding problems to be addressed are the following: (1) understanding the precise nature of the large N limit of the matrix model, and in particular the question of how the membrane topology is reflected in the matrix model; (2) the quantization of the (super-)membrane and the existence of the large N limit, and (3) the question of target space Lorentz symmetry, and how this symmetry is recovered in the large N limit in the quantum theory.

2 Seit 2008

2.1 Summary

Models from String Theory and other unified theories are closely interrelated with geometric partial differential equations of elliptic, parabolic and hyperbolic type. This opens up many new unforeseen and intriguing possibilities for exploiting the link between the mathematics of Ricci and Mean Curvature Flows on the one hand, and the physics of Einstein's equations and relativistic (super)membranes on the other. The first part of this project wants to utilize recent progress in analysis to understand the analytical and geometrical properties of solutions to specific models of theoretical physics such as p-branes in Lorentzian manifolds. During the first funding period the Cauchy problem for timelike minimal surfaces was solved in a very general setting, and the role of diffeomorphisms and gauge freedom for the solutions was clarified [1]. It became apparent that the structure of timelike minimal surfaces is intimately related to Einstein-Vacuum spacetimes on the one hand, but on the other hand also to analogous parabolic systems such as the Ricci flow and mean curvature flow as well as to the B-field models investigated in the project B4.

In a second, and closely related part of the project, the connections between membrane theories and $SU(N)$ matrix model approximation will be studied, in particular, the quantization of the (super)membrane and the question of Lorentz invariance and other target space symmetries. This part of the project builds on earlier work by H. Shimada [2,3] which was partially supported by this SFB.

2.2 Current Knowledge

2. Eigene Vorarbeiten

In his thesis [1] O. Milbredt investigated the Cauchy problem for submanifolds of a Lorentzian manifold which have an induced Lorentzian metric and vanishing mean curvature. Such submanifolds are often called timelike minimal surfaces and are used as models for the worldvolume of p-branes arising from the Nambu-Goto action in String Theory.

Milbredt proved the existence and uniqueness of shorttime solutions for given general space-like initial submanifolds and time-like initial tangent fields for the worldvolume. The method uses a harmonic gauge for the worldvolume to turn the vanishing mean curvature equation into a strictly hyperbolic quasilinear system. existence and uniqueness is then first shown in this particular gauge, a general geometric uniqueness result then follows from the construction of suitable diffeomorphisms from a more general gauge to the harmonic gauge. Crucial for the proofs is the deveolpment of suitable quantitative notions of "space-like" for the initial submanifold and "time-like" for the initial tangent-field to the worldvolume, which then lead to a priori estimates for the solutions in certain Sobolev-spaces. The work of Milbredt extends work by M65533;ller [4] on the worldsheet of strings, the shorttime existence in Minkowski space by Lindblad [5], stability of timelike minimal surfaces by Brendle [6], Xin [7], and existence results by Hoppe-Nicolai [8].
As for part two of the project, it has been known for a long time that the theory of relativistic membrane can be equivalently described as the $N \to \infty$ limit of certain $SU(N)$ matrix models [9,10].
This result can be used to set up a quantization scheme for the supermembrane. In [11] it was shown that for the unquantized supermembrane moving in a Minkowskian embedding space-time, Lorentz invariance in target space is recovered in the limit $N \to \infty$, and for totoidal membranes it was shown that the terms violating Lorentz invariance go like $1/N$ in this limit. More recent work of Shimada [2] (partially supported by this SFB) has established the precise link between membrane topology and the eigenvalue spectrum of the matrix theory. Exploiting these results, it should now be possible to extend previous results in this direction to membranes of arbitrary genus.

2.3 Methods

3. Forschungsprogramm

After the Cauchy problem for timelike minimal surfaces is solved in a general context, it is natural to ask for the longterm behaviour of solutions. Specific questions concern the existence and uniqueness of a maximal Cauchy development for given initial data and a characterisation of its boundary. Singularities are known to occur for general initial data, but little is known about their precise structure. In particular, it would be interesting to investigate the (in)stability of the membrane against formation of string-like "spikes" which is responsible for the continuity of the spectrum of the quantized supermembrane [12]. For the first part of the project it is planned to use techniques that have been fruitful in studies of the Einstein equations to find conditions that imply singularities and to find optimal gauges such as harmonic gauge or constant mean curvature gauge to analyze the behavior of solutions near singularities. It will also be interersting to investigate whether an additional B-field on the ambient manifold coupled to the vanishing mean curvature equation can serve to prevent certain singularities. Finally, following the work of Brendle [5], it is planned to investigate noncompact branes and their asymptotic decay behavior near infinity. A nontrival model case for such timelike minimal surfaces are surfaces close to the 2-dimensional catenoid in Euclidean 3-space, extended as a static solution of the time-like minimal surface equation to (3+1)-Minkowski space. It is hoped that the nonlinear stability of this static solution is somewhat easier to study than the nonlinear stability of the Schwarzschild solution in the case of the Einstein equations, while still exhibiting some of the nonlinear features of that problem.

The second part of the project will try to exploit the recent results obtained by Shimada and others. In particular, it will focus on the following questions:

- Approximating the PDE describing membrane evolution by a system of $N \times N$ matrix ODE's, can one re-obtain the PDE results derived previously (existence theorems, smoothness, etc.) by taking the limit $N \to \infty$, and for *arbitrary* topologies of the membrane and can one establish more quantitative estimates in this limit?

- Can one quantitatively study and describe the formation of singularities in the matrix approximation? And what would be the physical significance of such singularities, for instance, in application to the scattering of membranes? An important role here is played by topology changing singularities, for which the splitting and joining of strings in string scattering amplitudes a la Mandelstam are simple analogs.
- The matrix theory is known to become Lorentz invariant in the limit $N \to \infty$ at least for toroidal membranes [11]. Can one establish a similar result for arbitrary topology of the membrane, and furthermore study quantitatively how this limit is attained? Any progress in this direction would be an important step towards understanding the fate of Lorentz invariance for the quantized membrane, where Lorentz invariance could be broken by quantum effects (that is, anomalies, similar to the ones forcing the string to live in special 'critical dimensions').

2.4 Cooperations within the SFB

4. Vernetzung im SFB

The equations for mappings and submanifolds related to the B-field in project B4 are from an analytical point of view of the same quasilinear nature involving the Laplace-Beltrami operator as the equations for the membrane. In particular harmonic maps play again an essential role, so that a continued collaboration is expected. For example, it is planned to explore whether a B-field as considered in the PhD thesis of Koh in project B4 coupled to the membrane equation considered in this project can be adressed with the techniques of Milbredt and whether they lead to interesting new properties of solutions in the Lorentzian setting.

Further collaboration is expected with B3, where other nonlinear partial differential equations are studied that require similar analytical tools.

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