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1 [B5] Evolution of Geometrical Structures in Classical and Quantum Cosmology 2005-2008

1.1 Summary

The aim of project B5 is to explore the interface between cosmology and quantum gravity, in particular string theory. This motivates the extension of known techniques of mathematical cosmology to spacetimes of dimensions greater than four and the incorporation of matter fields which can be used to model cosmic acceleration (dark energy candidates). At the same time it leads to the examination of certain models in string theory in order to better understand their implications for cosmology.

1.2 Current Knowledge

In recent years a lot has been learned about the dynamics of solutions of the Einstein equations in classical general relativity, with coupling to various matter sources being included. Many things which were only known on the level of heuristic arguments or numerical calculations have been proved rigorously. An overview of these developments can be found in [R3]. In this context the solutions of Einstein's equations considered are in the spacetime dimension four corresponding to the three-dimensional space of everyday life. The matter fields typically considered are those of widespread applicability in astrophysics, such as perfect fluids, kinetic theory and Maxwell fields.

Work towards establishing a theory of quantum gravity, in particular in the area of string theory, has led to the consideration of spacetimes of higher dimensions and other types of matter fields. In [DHRW] a variety of matter fields in arbitrary dimensions motivated by string theory were studied mathematically from a certain point of view. The issue was the structure of general spacetime singularities and whether it is monotone or oscillatory. For a large class of cases where heuristic analyses indicate a monotone singularity it could be shown that there exists a class of solutions depending on the same number of free functions as the general solution where the singularity has the expected structure. This includes the case of the vacuum Einstein equations in all dimensions greater than ten.

Since the mid 1990's it has become increasingly clear, due to the accumulation of observational data, that the expansion of our universe is accelerated and this has become a central subject of interest in cosmology. Acceleration is only possible if there is either a positive cosmological constant or exotic matter with negative pressure (dark energy). Since there are some reasons for being unsatisfied with the solution of a cosmological constant alone, there has been a great deal of work in the literature on different candidate matter fields and their properties. At the moment there is no clearly preferred choice and it makes sense to consider a wide range of possibilities. This research gives rise to a number of interesting mathematical problems. More information on this can be found in [R5]. Here there is a clear opportunity for a symbiosis between cosmology and quantum gravity. On the one hand cosmologists are looking for new models for dark energy. On the other hand cosmological observations have the potential to provide the first empirical tests of theories of quantum gravity.

A natural starting point for the mathematical study of dark energy models, both from the observational and theoretical point of view, is that of homogeneous models. Starting from a classical result of Wald [W] on spacetimes with positive cosmological constant, theorems were proved on the late-time behaviour of solutions of the Einstein equations in the presence of dark energy [R2], [L], [R4]. In the fully inhomogeneous case nothing about the late-time behaviour of general cosmological solutions was known except for a result of Friedrich [F] in the case of a positive cosmological constant. The existence of solutions in arbitrary dimension depending on the same number of free functions as the general solution and having late-time accelerated expansion was obtained in [R1]. In that paper a complete rigorous analysis of the vacuum case in any dimension on the level of formal series was given following the heuristic work of Starobinsky [S] in four dimensions. It is interesting that essentially the same expansions as can be found in [S] were developed independently slightly later by Fefferman and Graham FG]. This is an example of the same mathematical object being studied in astrophysics and mathematics with the people concerned and apparently also those who quoted them in the ensuing twenty years being unaware of this fact.

1.3 Results and their Interpretation

The research of project [B5] so far has led to results extending the current knowledge discussed above in a number of directions. The dynamics of cosmological solutions of the Einstein equations in dimensions greater than four was studied. In four dimensions it was investigated from a mathematical point of view how the presence of exotic matter fields affects the evolution of solutions of the Einstein equations. Going further, results were obtained on some alternative theories of gravity. In particular, the so-called Cardassian models were studied. A topic of research on the string theory side was that of superstring amplitudes in the hybrid formalism.

1.3.1 Higher dimensional models

One obvious challenge is to generalize the result of [F] to higher dimensions. It is not straightforward to do this since the proof of [F] uses techniques of conformal geometry specific to four dimensions. More recently a generalization to higher even dimensions was found by Anderson [A]. It uses different conformal techniques, in particular the obstruction tensor of Fefferman and Graham [FG]. The restriction to even dimensions has to do with the fact that the asymptotic expansions which describe the long-time behaviour of solutions generally contain logarithmic corrections in odd dimensions. Together with Mark Heinzle, Alan Rendall was able to make a different kind of contribution [HR] to this aspect of project [B5]. We used the fact that the Einstein equations coupled to certain nonlinear scalar fields in four dimensions can be related directly to solutions of the Einstein vacuum equations in higher dimensions. In this way the results of [A] were used to prove an analogue of the theorem of [F] for four-dimensional spacetimes with dark energy described by a nonlinear scalar field (quintessence field). Thus in this case the flow of information was from higher dimensions to four dimensions. In more detail, the result is as follows. If a solution of the Einstein vacuum equations with positive cosmological constant in $4+d$ dimensions admits an action of the torus T^d by isometries and possesses an additional reflection symmetry it is possible to show, by Kaluza-Klein reduction, that it can be related to a solution of the Einstein equations coupled to a nonlinear scalar field ϕ in four dimensions. The four-dimensional metric is related by a conformal rescaling to the quotient of the $4+d$ dimensional metric by the symmetry group. The scalar field is related to the volume of the group orbits and also defines the conformal factor. The potential of the scalar field is exponential with exponent $-\sqrt{\frac{2d}{d+2}}$.

1.3.2 Exotic matter models

The known results for homogeneous models with a cosmological constant or minimally coupled scalar field were generalized in [Re6] to some of the k-essence models where the Lagrangian $-\frac{1}{2}\nabla_\alpha\nabla^\alpha - V(\phi)$ of a minimally coupled field is replaced by a more general expression of the form $L(\phi, \nabla_\alpha\phi\nabla^\alpha\phi)$. These Lagrangians have been applied in cosmology and can be related to low-energy expansions in string theory.

Another type of matter model which comes up in expansions of models in quantum field theory is that of curvature coupled scalar fields. In minimal coupling the Lagrangian contains no derivatives of the spacetime metric of order higher than one. In the curvature-coupled case it contains the expression $\xi R\phi^2$ where R is the scalar curvature and ξ is a coupling constant. This can lead to accelerated expansion for scalar fields with potentials which do not have that effect in the case $\xi = 0$ of minimal coupling. In his thesis, done under the supervision of Alan Rendall, Roger Bieli has investigated different aspects of the late-time behaviour of spacetimes with a curvature-coupled scalar field. He has been able to generalize many of the results previously obtained in the minimally coupled case and identify important differences between the two cases. This has already resulted in two publications [B1] and [B2]. Among other things the results include a complete analysis of late-time behaviour on the level of formal series and results on the late-time asymptotics for interesting classes of potentials in the homogeneous case.

1.3.3 Alternative theories of gravity

Another way in which non-minimal coupling can come into cosmology is through a Lagrangian for the gravitational field involving the curvature tensor and possibly its derivatives. The simplest class of models of this type are the $f(R)$ theories. In that case the scalar curvature R, which defines the Einstein-Hilbert Lagrangian of general relativity, is replaced by the expression $f(R)$ for some smooth function f. For suitable choices of f this can lead to accelerated expansion without any matter fields being necessary. For non-trivial choices of f the equations of motion of the gravitational field are fourth order, in contrast to the Einstein equations which are second order. This leads to major difficulties in the mathematical treatment. For the $f(R)$ theories, in contrast to more general gravitational Lagrangians involving the curvature, this problem can be got around by a trick. Introducing a certain scalar field and using it to conformally transform the metric reveals that the field equations of $f(R)$ gravity are equivalent to the Einstein equations coupled to a minimally coupled nonlinear scalar field with potential V. The function V can be computed from f. Alan Rendall has already used this device to obtain statements on late-time asymptotics for $f(R)$ theories of gravity [R7]. This is being carried further by Lucy MacNay.

Another type of alternative theories is defined by the Cardassian models, at least in the homogeneous and isotropic case. These can be related to different models in quantum gravity, namely Loop Quantum Gravity and braneworld models. In the early stages of project [B5] useful input on these matters was provided by Martin Bojowald. In work for his diploma, under the supervision of Jan Plefka and Alan Rendall, Nikolaus Berndt carried out a rigorous study of the dynamics of solutions of these theories in the isotropic case, which is the one considered almost exclusively in the literature. These solutions show finite-time singularities, so-called big rip singularities, in many cases. Berndt showed that there is a natural way of generalizing Cardassian models to the general homogeneous case and then showed rigorously that big rip singularities are associated with isotropization, a phenomenon which has close similarities with the theorem of Wald mentioned earlier. His further investigations raise serious doubts whether there is any reasonable generalization of Cardassian models to the inhomogeneous case.

1.3.4 String Theoretical Aspects

Jan Plefka joined the B5 project in autumn of 2006. Unfortunately the postdoc hired soon thereafter from the SFB funds, Jürg Käppeli, left already in February 2007 for a consultant job in Zürich. In his four months at Humboldt-University Käppeli finished a paper on the computation of superstring scattering amplitudes in the hybrid formalism due to Berkovits [Kap]. Here the structure of Weyl tensor squared terms in the four dimensional effective action of Calabi-Yau compactifications of the superstring were established.

Within the proposal by Plefka it was planned to study nonperturbative descriptions of string theory in time dependent backgrounds in form of matrix models in the large N limit. Clearly progress here was hindered due to the unexpected leave of Käppeli. In October 2007 Subodh Patil took up the vacant position and has enforced the Humboldt group with string cosmological expertise. The proposed computation of the effective potential of the matrix big bang model of Craps et al for non-static backgrounds will now be pursued . In a very recent paper [Pathil1] cosmological consequences of a mechanism put forward by Dvali et. al. for solving the cosmological constant problem in the context of a theory of massive (resonance) gravitons was studied. In this framework, Newton's constant gets promoted to a function of the covariant D'Alembertian in such a way that constant spacetime sources degravitate. It is shown that slow roll inflation is preserved in this setup and that the non-local effects of degravitation make the mechanism sensitive to global features of the universe. Moreover the intriguing possiblity is pointed out that even though degravitation completely kills the cosmological constant, the presently observed vacuum energy density might be due to an 'afterglow' of previous false vacua.

In a second line of research the consequences of UV/IR mode mixing in the context of non-commutative field theory in a minimally extended model of non-commutative inflation are presently explored. A paper should appear soon.

1.4 Relations within the SFB

Many of the results obtained in this project are results on quasilinear wave equations and thus related to project [A4]. Cosmology as a field of application of nonlinear wave equations is a subject of broad interest within the SFB and with this in mind Lars Andersson, Hermann Nicolai, Jan Plefka, Alan Rendall and Stefan Theisen organized a seminar on cosmology which was held at the Humboldt University and the AEI. The aim was to better understand the book 'Physical foundations of cosmology' by V. Mukhanov while sharing the complementary knowledge of different participants on the subject. The central subject was the theory of microwave background fluctuations which, from a mathematical point of view, is based on an understanding of the global dynamics of solutions of quasilinear wave equations. In September 2007 Lars Andersson, Mihalis Dafermos, Alan Rendall and Igor Rodnianski organized an international conference entitled 'Evolution equations and self-gravitating systems' at AEI.

Other results of the project, in particular those concerning ordinary differential equations, can usefully be seen in the context of the theory of dynamical systems and are hence closely connected to project [B3]. Motivated by this, some of the meetings of the group seminar of Bernold Fiedler were used to discuss dynamical systems in cosmology, with extended talks by Alan Rendall and Stefan Liebscher. The goal was to explain mathematical problems of interest in cosmology to the experts in dynamical systems and explore the prospects for applying advanced techniques which have not yet penetrated the literature on mathematical relativity.

It is intriguing that the Q-curvature, which plays a role in project [A6] is related to the work of Fefferman and Graham [FG] quoted above. It is to be hoped that, if the right connection could be found, there could be a valuable collaboration between these two projects.

1.5 Relations to other Research Work

The work of Anderson [A] already mentioned has been generalized and extended to other situations (asymptotically flat spacetimes) by Anderson and Chrusciel [AC]. Another paper involving both Kaluza-Klein reduction and accelerated expansion is [AH]. The work on k-essence models in [R6] led to a fruitful interaction with Alexander Vikman, who together with others has been heavily involved with such models (see e.g. [BMV]). There is a fast growing recent literature on $f(R)$ theories. For some references see [M]. The Cardassian models were introduced in [FL]. Despite the limitations on the range of the definition, as mentioned above, this paper has already collected over 200 citations on the astrophysics archive.

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