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1 [B6] Analytic and Spectral Properties of Geometric Operators 2005-2008

1.1 Summary

In project B6 we study relations between analytic properties of geometrically and physically relevant differential operators and the geometry of the underlying manifold. The questions under consideration include analytic properties of operators on singular and possibly incomplete manifolds as well as inverse spectral problems.

1.2 Results and their Interpretation

Among the results obtained so far are path integral representations of solutions to the heat equation on compact manifolds and spectral approximation facts for incomplete hyperbolic 3-manifolds. We have studied overdetermined equations such as twistor-spinors on orbifolds and transversal Killing spinors. This links our work to project A2. We have studied the solution space to the Seiberg-Witten equations from a geometric perspective.

Moreover, we have obtained several inverse spectral results: "Positive" results concerning spectral rigidity properties of biinvariant metrics and spectral determination of orientability of closed hyperbolic surfaces, as well as "negative" inverse spectral results concerning spectral nondetermination of total integrability of geodesic flows and of the maximal order of singularities in compact orbifolds.

Path integrals on manifolds.

Let M be a compact Riemannian manifold without boundary, let $E \to M$ be a Riemannian or Hermitian vector bundle with compatible connection ∇. Let H be a selfadjoint generalized Laplace operator, i. e. an operator of the form $H = \nabla^*\nabla + V$ where V is a potential (symmetric endomorphism field on E).

The main result of [BP08] can formally be stated as follows: The solution to the heat equation

$$\frac{\partial U}{\partial t} + HU = 0$$

with initial condition

$$U(0,x) = u(x)$$

is given by the path integral

$$U(t,x) = \frac{1}{Z}\int_{\mathcal{C}_x(M,t)} \exp\left(-\frac{1}{2}\mathrm{E}(\gamma) + \int_0^t \left(\frac{1}{3}\mathrm{scal}(\gamma(s)) - V(\gamma(s))\right) ds\right) \cdot \tau(\gamma)_t^0 \cdot u(\gamma(t))\, \mathcal{D}\gamma.$$

Here $\mathcal{C}_x(M,t)$ is the space of all continuous paths $\gamma : [0,t] \to M$ emanating from x, $\mathrm{E}(\gamma)$ denotes the energy of the path γ, $\tau(\gamma)$ is parallel translation along γ, $\mathcal{D}\gamma$ is a formal measure on $\mathcal{C}_x(M,t)$ and Z is a normalizing constant.

Such formulas are very common in the physics literature but there are various problems with a rigorous mathematical interpretation:

- $\mathcal{C}_x(M,t)$ is an infinite dimensional space and the meaning of the measure $\mathcal{D}\gamma$ is unclear,
- $\mathrm{E}(\gamma)$ and $\tau(\gamma)$ are not defined for continuous paths without differentiability properties,
- $\frac{1}{Z}$ is infinite.

It is well-known that $\frac{1}{Z}\exp\left(-\frac{1}{2}\mathrm{E}(\gamma)\right)\mathcal{D}\gamma$ yields a well-defined measure on path space $\mathcal{C}_x(M,t)$, the *Wiener measure*. Parallel transport $\tau(\gamma)$ can be treated using stochastic differential equations. This then generalizes the *Feynman-Kac formula*, see e. g. [DT01].

In [BP08] one follows a different approach which does not make use of any stochastics. One approximates $\mathcal{C}_x(M,t)$ by finite dimensional spaces of geodesic polygons. It turns out that the formally identical integrals over these finite dimensional spaces approximate the solution to the heat equation. The necessary analysis can be organized nicely using a classical theorem of Chernoff's [Che68]. The short time asymptotics of the heat kernel also play an important role.

This technique allows one to derive different versions of the path integral formula. For example, one can remove the scalar curvature term if one uses another measure on the approximating spaces of geodesic polygons. This clarifies a discussion in [AD99] where the path integral formula has been proved by different methods in the special case of the Laplace-Beltrami operator acting on functions.

As an application one finds a very simple and natural proof of the Hess-Schrader-Uhlenbrock estimate for the heat kernel by the kernel of a scalar comparison operator, see [HSU80]. Moreover, one can express the trace of the heat operators by a path integral. Formally,

$$\begin{aligned}\mathrm{Tr}(e^{-tH}) \quad &= \\ &\frac{1}{Z}\int_{\mathcal{C}_{cl}(M,t)} \exp\left(-\frac{1}{2}\mathrm{E}(\gamma) + \int_0^t \left(\frac{1}{3}\mathrm{scal}(\gamma(s)) - V(\gamma(s))\right) ds\right) \mathrm{tr}(\mathrm{hol}(\gamma))\, \mathcal{D}\gamma.\end{aligned}$$

Here $\mathcal{C}_{cl}(M,t)$ denotes the space of closed continuous loops in M, parametrized on $[0,t]$, and $\mathrm{hol}(\gamma)$ is the holonomy of such a loop γ. One might hope that this formula will yield a new proof of the Atiyah-Singer index theorem using ideas from the physics literature [Wit99].

Further open problems concern path integral formulas in the presence of a boundary (e. g. with Dirichlet boundary conditions) or for the Friedrichs extension on a possibly incomplete manifold.

The Laplacian on incomplete hyperbolic 3-manifolds.

Let M_∞ be a complete non-compact hyperbolic 3-manifold of finite volume, and let $\mathrm{Def}(M_\infty)$ denote the deformation space of (possibly incomplete) hyperbolic structures on M_∞, for a precise definition see [CHK00]. These incomplete structures are said to have *Dehn surgery type singularities*, special cases include hyperbolic cone-manifold structures and in particular smooth hyperbolic structures on certain topological fillings.

Now, $\mathrm{Def}(M_\infty)$ carries a natural topology, and by Thurston's Cusp Closing Theorem (see [Thu78]) there are sequences $(M_n)_n$ of closed hyperbolic

3-manifolds such that $M_n \to M_\infty$ as $n \to \infty$ in $\mathrm{Def}(M_\infty)$ (after one has removed certain closed geodesics from the M_n).

On M_∞ the Laplacian Δ^{M_∞} is essentially self-adjoint, and its spectrum has the form $\mathrm{spec}\left(\Delta^{M_\infty}\right) = \{0 = \lambda_0 \leq \lambda_1 \leq \dots \lambda_k < 1\} \quad \dot{\cup} \quad [1, \infty)$ (see [HD79] and [LP82]).

For $M \in \mathrm{Def}(M_\infty)$ we consider the Friedrichs extension of $\Delta : C_c^\infty(M) \to L^2(M)$ and we denote it by Δ^M. As the ends of M are foliated by tori an argument involving separation of variables (similar to an argument in [Bär00]) shows that Δ^M has discrete spectrum.

Now, we are interested in the behavior of $\mathrm{spec}(\Delta^M)$ as $M \to M_\infty$ in $\mathrm{Def}(M_\infty)$. Answers to this question have already been given in [BC91] and [CD94] in the case that M_∞ is approximated by closed hyperbolic 3-manifolds as in the Cusp Closing Theorem. In [PW07] these results are generalized to the situation when $M \to M_\infty$ in $\mathrm{Def}(M_\infty)$. One obtains that the small eigenvalues (i. e. those below 1) of M converge to the small eigenvalues of M_∞, one gets a clustering of the eigenvalues above 1 of the M and one computes the accumulation rate in terms of geometric data.

A singularity theorem for twistor-spinors.

Let M^n be an n-dimensional Riemannian *orbifold*. Assume it to be spin: for orbifolds with singularities of codimension greater than 2 this is equivalent to their smooth part being spin (this is the first preliminary result of [BGR07]). Fix the spin structure and let ΣM be its spinor bundle, "$\cdot$" its Clifford multiplication, ∇^M the canonical covariant derivative on ΣM and D its Dirac operator.

A *twistor-spinor* on M is a smooth section ψ of ΣM satisfying

$$\nabla_X^M \psi = -\frac{1}{n} X \cdot D\psi$$

for all $X \in TM$. We aim at understanding the geometric conditions imposed by the existence of such a spinor field on M. The main result of [BGR07] may be stated as follows:

Assume M^n compact, $n \geq 3$ and that a non-zero twistor-spinor ψ exists on M^n with zero in p. Then the following holds:

1. *The zero-set of ψ is reduced to $\{p\}$ and p is singular unless M^n is conformally equivalent to the round sphere S^n.*

2. *For every point $q \neq p$ one has*

$$|\Gamma_q| \leq |\Gamma_p|,$$

where Γ_q is the singularity group of q in M^n.

Moreover $|\Gamma_q| = |\Gamma_p|$ for some $q \neq p$ only if M^n is conformally equivalent to a quotient of S^n through a finite subgroup Γ of SO_{n+1}.

Transversal Killing spinors.

Let (M, g, ξ) be an $n+1$-dimensional spin *Riemannian flow* (Riemannian manifold endowed with a unit smooth vector field ξ satisfying the Killing-type equation $g(\nabla^M_X \xi, Y) = -g(\nabla^M_Y \xi, X)$ for all $X, Y \in \xi^\perp$). Let ΣM denote its spinor bundle, "$\cdot$" its Clifford multiplication, ∇ the *transversal* covariant derivative on ΣM and D its Dirac operator. The covariant derivative ∇ may be defined by the following relations:

$$\begin{aligned} \nabla^M_\xi \varphi &= \nabla_\xi \varphi + \tfrac{1}{2}\Omega \cdot \varphi + \tfrac{1}{2}\xi \cdot \nabla^M_\xi \xi \cdot \varphi \\ \nabla^M_Z \varphi &= \nabla_Z \varphi + \tfrac{1}{2}\xi \cdot \nabla^M_Z \xi \cdot \varphi. \end{aligned}$$

for all $Z \in \xi^\perp$, where the 2-form Ω on $\xi^\perp$ is defined by $\Omega(Y, Z) := g(\nabla^M_Y \xi, Z)$ and ∇^M denotes the canonical covariant derivative on TM or ΣM.

For real α, β we call (α, β)-*transversal Killing spinor* on (M, g, ξ) any smooth section ψ of ΣM satisfying

$$\nabla_X \psi = \alpha g(X, \xi)\xi \cdot \psi + \beta\, \xi \cdot X \cdot \psi + \beta g(X, \xi)\psi$$

for all $X \in TM$. Our aim is two-fold: on the one hand we want to understand which geometries support transversal Killing spinors, on the other hand we want to derive upper eigenvalue estimates by testing the min-max principle on those spinors, in order to compare the result with O. Hijazi's lower bound [Hij95] involving the energy-momentum tensor. One of the motivations for this work consists in looking for examples where this lower bound is sharp but not T. Friedrich's one [Fri80] in terms of the scalar curvature solely.

The main spectral result [GH07c] in the setting of Sasakian manifolds (see [GH07b, GH07a] for general as well as 3-dimensional flows) may be stated as follows:

Assume the existence of a non-zero (α, β)-transversal Killing spinor ψ on a $2m+1$-dimensional compact Sasakian *spin manifold (M, g, ξ).*

1. *The smallest eigenvalue $\lambda_1(D^2)$ of D^2 satisfies*

$$\lambda_1(D^2) \leq \alpha^2 + 4m^2\beta^2 + \frac{m^2}{4} + \alpha\frac{\int_M \langle \xi \cdot \Omega \cdot \psi, \psi\rangle v_g}{\mathrm{Vol}(M)}.$$

2. *The equality can only occur if $\beta = 0$. In that case, the Sasakian manifold (M^{2m+1}, g, ξ) is η-Einstein and if moreover $\alpha \neq 0$ then*

$$\lambda_1(D_M^2) = \inf_{M\setminus Z_\phi}\left(\frac{\mathrm{Scal}_M}{4} + |E^\phi|^2\right),$$

where $Z_\phi := \{x \in M \,|\, \phi_x = 0\}$ and E^ϕ is the 2-tensor field defined on $M \setminus Z_\phi$ by $E^\phi(X,Y) := \Re(\langle Y \cdot \nabla_X^M \phi, \frac{\phi}{|\phi|^2}\rangle)$ for all $X, Y \in \Gamma(TM)$. In particular O. Hijazi's estimate [Hij95] must be sharp.

The Dirac spectrum of $\mathrm{SU}(2)/Q_8$.

Let $M := \mathrm{SU}(2)/\mathrm{Q}_8$, where Q_8 denotes the (finite) group of quaternions. It is a 3-dimensional spin compact connected homogeneous space carrying a 3-parameter-family of homogeneous Riemannian metrics and 4 different spin structures. We aim at computing the spectrum - or at least the smallest eigenvalue - of the Dirac operator for the so-called Berger metrics. The motivation for this work comes from the study of the limiting-case of C. Bär's upper bound [Bär98] (see below) in terms of the mean curvature for hypersurfaces in spaceforms. Indeed M can be naturally embedded in S^4 and constitutes the simplest example after geodesic spheres and generalized Clifford tori of hypersurfaces in S^n where both the mean curvature and the Dirac operator can be computed. The main result of [Gin08] dealing with this question can be stated as follows:

The smallest eigenvalue $\lambda_1(D^2)$ of the Dirac Laplacian of M equals $\frac{9}{4}$ for any minimal embedding $M \hookrightarrow S^4$ and w.r.t. the induced spin structure. In particular any such embedding satisfies the equality in the following estimate due to C. Bär [Bär98]:

$$\lambda_1(D^2) \leq \frac{9}{4\mathrm{Vol}(M)}\int_M (H^2+1)v_g,$$

where H denotes the mean curvature.

The geometry of Seiberg-Witten moduli spaces.

Let (M,g) be a compact Riemannian 4-manifold with a fixed Spin^C-structure P. The perturbed Seiberg-Witten equations are coupled nonlinear elliptic equations for a connection $A \in \mathcal{A}(\det P)$ and a positive spinor $\psi \in \Gamma(\Sigma^+)$, perturbed by a parameter $\mu^+ \in \Omega^2_+(M; iR)$. The Seiberg-Witten premoduli space $\widetilde{\mathcal{M}}_{\mu^+} \subset \mathcal{A}(\det P) \times \Gamma(\Sigma^+)$ is the solution space of those equations. The Seiberg-Witten moduli space, which is the quotient of $\tilde{\mathcal{M}}_{\mu^+}$ by the action of the gauge group $\mathcal{G} = \mathcal{C}^\infty(M; U(1))$, is well known to be a compact, generically smooth manifold. The Seiberg-Witten bundle $\mathcal{P} \to \mathcal{M}_{\mu^+}$ is an isomorphism class of principal $U(1)$ bundles on $\mathcal{M}_{\mu^+}$, represented by the quotient of $\tilde{\mathcal{M}}_{\mu^+}$ by the based gauge group

$$\mathcal{G}_{x_0} := \{u \in \mathcal{C}^\infty(M; U(1)) \,|\, x(x_0) = 1\},$$

$x_0 \in M$ being arbitrary. The Seiberg-Witten invariant is a differential-topological invariant of $\mathcal{M}_{\mu^+}$, given by evaluation of the Euler class of $\mathcal{P}$ on the fundamental cycle of $\mathcal{M}_{\mu^+}$. For generic smooth families of perturbations, the parametrized moduli space $\mathcal{N}_{\mu^+} := \bigsqcup_{t\in[0,1]} \mathcal{M}_{\mu^+(t)}$ is also a smooth manifold.

The aim of our work was to refine the knowledge about the moduli space $\mathcal{M}_{\mu^+}$ from only differential-topological to geometrical properties. In [Bec07], natural constructions for Riemannian metrics on the spaces $\mathcal{M}_{\mu^+}, \mathcal{P}$ and $\mathcal{N}_{\mu^+}$ are given: The infinite dimensional premoduli space $\tilde{\mathcal{M}}_{\mu^+}$ naturally inherits a weak Riemannian metric from the L^2-metric on the configuration space $\mathcal{C} = \mathcal{A}(\det P) \times \Gamma(\Sigma^+)$. As a quotient of $\tilde{\mathcal{M}}_{\mu^+}$, the Seiberg-Witten moduli space can be given a quotient metric, which is indeed a Riemannian metric. To obtain similar constructions for Riemannian metrics on $\mathcal{P}$ and $\mathcal{N}_{\mu^+}$, these spaces need to be represented as good quotients from subspaces of the configuration space $\mathcal{C}$. E.g. for $\pi_1(M) = 0$, the isomorphism class $\mathcal{P} \to \mathcal{M}_{\mu^+}$ is shown to have a natural geometric representative, the total space of which is a quotient of a subspace of $\mathcal{C}$ admitting a quotient metric.

After giving constructions for those representations, it is shown, that the Riemannian metrics constructed on $\mathcal{M}_{\mu^+}, \mathcal{P}$ and $\mathcal{N}_{\mu^+}$ fit nicely together: The bundle projection $\pi : \mathcal{P} \to \mathcal{M}_{\mu^+}$ is a Riemannian submersion. The metric induced by the inclusion of the slice $t = t_0$ in $\mathcal{N}_{\mu^+}$ coincides with the metric constructed on $\mathcal{M}_{\mu^+(t_0)}$, provided the slice is a smooth manifold.

In the case of a Kähler manifold (M,g), Seiberg-Witten monopoles are well known to be representable in terms of holomorphic data. For perturbations $\mu^+ = i\pi\lambda \cdot \omega$ along the Kähler form, these representations yield to the identification of the moduli space $\mathcal{M}_{\mu^+}$ as a CP^n-fibration over the torus $H^1(M; iR)/H^1(M; 2\pi iZ)$.

For manifolds with $b_2^+(M) > 1$, the parametrized moduli space $\mathcal{N}_{\mu^+}$ is generically a smooth cobordism between the moduli spaces $\mathcal{M}_{\mu^+}$ for different values of the perturbation μ^+. In case $b_2^+(M) = 1$, it has conical singularities along the wall of those perturbations admitting reducible solutions to the Seiberg-Witten equations. As is well known e.g. from work by Okonek and Teleman [CO96], if $b_2^+(M) = 1$ and $b_1(M) = 0$, the moduli spaces $\mathcal{M}_{\mu^+}$ collaps to a point as the perturbation approaches the wall.

In [Bec07], it is shown that this collaps indeed occurs in the intrinsic Riemannian metrics of the slices of the cobordism $\mathcal{N}_{\mu^+}$; i.e. the diameter of the slices tends to 0 as the perturbation approaches the wall.

Finally, the main result of [Bec07] states, that on a Kähler surface (M, g), our quotient L^2-metric on the regular part of the moduli space $\mathcal{M}_{\mu^+}$ is itself a Kähler metric. This is proven by means of infinite dimensional generalizations of the symplectic resp. kählerian reduction, following similar work by Hitchin [Hit86]. Symmetry arguments imply, that for $M = CP^2$ with the Fubini-Study metric and $P = P_0 \otimes \mathcal{O}(1)$, the quotient L^2-metric on $\mathcal{M}_{\mu^+} = CP^2$ is the Fubini-Study metric.

Local spectral rigidity of biinvariant metrics.

An important open conjecture in spectral geometry says that a symmetric space of compact type is uniquely determined by the spectrum of the associated Laplace operator on functions. A special case that has been studied extensively is that of the round sphere. By using heat invariants, S. Tanno verified the conjecture for standard spheres in dimension six and lower, and demonstrated in 1980 that in all dimensions, the standard metric is locally spectrally determined within the class of all metrics on S^n [Tan80].

A natural class of symmetric spaces are compact Lie groups equipped with a biinvariant metric. D. Schüth had shown in [Sch01] that there do not exist non-trivial continuous isospectral deformations of a biinvariant metric within the class of left invariant metrics on a compact Lie group. This infinitesimal rigidity result leaves open the question of whether a biinvariant metric might be locally spectrally determined within the class of left invariant metrics. In collaboration with C.S. Gordon and C.J. Sutton we have proved that this is indeed the case [GSS10]. Hence, within the class of left invariant metrics on a compact Lie group G, any metric $g \neq g_0$ that is isospectral to a biinvariant metric g_0 must be sufficiently far away from g_0. (In contrast, there do exist continuous isospectral families of left invariant metrics [Sch01].) In the special case of compact simple Lie groups we have obtained a still stronger local rigidity result: The biinvariant metric on a simple compact Lie group is locally determined among left invariant metrics of at most the same volume by the first two nonzero eigenvalues $0 < \lambda_1 < \lambda_2$ (ignoring multiplicities) of the associated Laplace operator.

Spectrum and orientability.

We recall that the spectrum of the Laplace operator on a compact Riemannian manifold determines, among other invariants, its total scalar curvature. In the case of closed oriented surfaces, the Gauss Bonnet formula thus implies that the spectrum determines the topology of the surface. Intriguingly, however, it is an open question whether the spectrum determines orientability in the first place – for surfaces as well as for Riemannian manifolds of arbitrary dimension. In the case of surfaces with boundary, P. Bérard and D.L. Webb had given examples of Neumann isospectral flat surfaces with boundary, one of which is orientable and the other not [BW95]. In collaboration with P.G. Doyle, J.P. Rossetti proved that in the case of closed hyperbolic surfaces, the spectrum of the Laplace operator on functions does indeed determine whether the surface is orientable or not [DR08]. The proof extensively uses the Selberg trace formula and estimates on the growth of the length spectrum of a compact hyperbolic surface.

Spectrum and geodesic flows.

The singularities of the wave trace on a Riemannian manifold are contained in the set of lengths of closed geodesics on the manifold. Asymptotic expansions of these singularities near such a length yield, under suitable nondegeneracy assumptions, geometric information on the set of closed geodesics of this length; see, e.g., the foundational article by H. Duistermaat and V. Guillemin [HD75] or the survey article [Zel04] by S. Zelditch.

Closed geodesics thus being at the focus of the wave invariants, it is natural to ask to which extent integrability properties of the dynamical system given by the geodesic flow of a Riemannian manifold are determined by spectral data. In particular, it was an open problem whether complete integrability of the geodesic flow in the sense of Liouville is a property determined by the Laplace spectrum on functions. We have proved in [Sch08] that this is *not* the case: There exists a pair of compact closed isospectral Riemannian manifolds M, M' such that M has completely integrable geodesic flow, while M' does not have completely integrable geodesic flow. More precisely, the manifolds M and M' in our counterexample are compact, eight-dimensional, two-step Riemannian nilmanifolds. For both manifolds M and M' we have also analyzed the structure of the submanifolds of the unit tangent bundle given by maximal continuous families of closed geodesics with generic velocity fields. The structure of these submanifolds turns out to reflect the above (non)integrability properties. On the other hand, their dimension is larger than that of the Lagrangian tori in M, indicating a degeneracy which might explain the fact that the wave invariants do not distinguish an integrable ¿from a nonintegrable system here. Finally, we have shown that for M, the invariant eight-dimensional tori which are foliated by closed geodesics are dense in the unit tangent bundle, and that both M and M' satisfy the so-called Clean Intersection Hypothesis.

Singularities of isospectral orbifolds.

To which extent does the Laplace spectrum determine the geometry of a compact Riemannian orbifold, and, in particular, the structure of its singularities? There exist some positive results in this direction. E. Dryden and A. Strohmaier showed that on oriented compact hyperbolic orbifolds in dimension two, the spectrum completely determines the types and numbers of singular points [DS05]. By a result of E. Stanhope, only finitely many isotropy groups can occur in a family of isospectral orbifolds satisfying a uniform lower bound on the Ricci curvature [Sta05]. On the other hand, there exist arbitrarily large (finite) families of mutually isospectral Riemannian orbifolds such that each of these contains an isotropy group which does not occur in any of the other orbifolds of the family [NS06]. More precisely, for the maximal isotropy orders occurring in the orbifolds of such a family, the corresponding isotropy groups all have the same order, but are mutually nonisomorphic. A natural question arising in this context is whether the spectrum might nevertheless determine the size of the maximal isotropy groups. J.P. Rossetti, D. Schüth, and M. Weilandt have succeeded in showing that this is *not* the case; see [RSW08]. In that paper, we give several kinds of examples of isospectral connected orbifolds with different maximal isotropy orders. Some of these arise from a version of the famous Sunada theorem and are orbifolds quotients of suitable normal homogeneous spaces; other examples belong to the category of flat orbifolds and are shown to be isospectral by explicit computation, using formulas developed earlier by R. Miatello and J.P. Rossetti. In the latter type of examples, the orbifolds are not isospectral on 1-forms.

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2.1 Expected Results and their Intepretation

Despite the many results obtained a lot of questions remain open. The technique developed in [BP08] to derive path integral formulas has turned to be quite flexible and powerful. We plan to elaborate on it to shed light on the following questions:

- In the case of manifolds with boundary how are boundary conditions reflected in the path integral?
- How does a path integral formula look like for the Friedrichs extension on an incomplete manifold?
- Is it possible to derive the Atiyah-Singer index theorem from the path integral for the trace of the heat operator?
- Can one make the physicist's "Wick rotation" rigorous and obtain a path integral for the solution to the Schrödinger equation in this geometric context?

Here path integrals are always to be understood in the sense of finite dimensional approximation as in [BP08]. A rather ambitious question would be if one can define an "integral over surfaces" rather than paths by looking a triangulations of the surface whose mesh tends to zero. An affirmative answer to this question would be an important breakthrough since it would allow to make path integrals in string theory rigorous where they are often used on a heuristic level. In this situation stochastic techniques are not available. One would then have to check how such definitions of the path integral relate to competing definitions from spectral geometry such as the ones based on regularized determinants, see e.g. [BS03].

It is also planned to continue the investigation of the analysis of geometric operators on incomplete manifolds. Incomplete manifolds often arise as the regular part of a singular space. Classical work on this topic (e. g. [Che83] or the work of the schools of Melrose and of Schulze) typically makes rather rigid structural assumptions on the space like being an orbifold or a cone manifold. We will try to replace this by softer assumptions on geometric quantities like isoperimetric constants in order to show properties like essential self-adjointness, Fredholmness etc.

In project B4 important progress was obtained on the harmonic map flow coupled to a Deligne cohomology class (or gerbe). If the class has degree two this describes magnetic geodesics, in degree three one has coupled the kinetic energy to what is known in string theory as a B-field. It turned out that a better geometric understanding of concepts like fiber integration in Deligne cohomology or covariant derivatives of sections in Deligne classes is necessary. It is planned to work this out.

In the area of inverse spectral geometry, we plan to investigate several other questions which have remained open. In the context of [GSS10] we will try to extend our local spectral rigidity result which we obtained there for biinvariant metrics. More precisely, we will study the questions whether the metric on a (locally) symmetric space of compact type is infinitesimally (or even locally, or even globally) spectrally rigid within homogeneous (or even within arbitrary) Riemannian metrics on the manifold. Local symmetry is a property of the Riemannian curvature tensor. We will also consider other questions concerning the relations between the spectrum and special curvature properties: Are properties like, say, the Einstein condition, or local harmonicity, or the d'Atri property spectrally determined? Is the isomorphism class of the connected holonomy group spectrally determined?

We plan to continue our investigations of the spectral geometry of orbifolds: Which information about the singularities of a compact Riemannian orbifold is (not) encoded in the spectrum on functions or differential forms? In the context of the results of [RSW08], some immediate questions which have remained open are the following: Do there exist pairs of spherical or of hyperbolic Riemannian orbifolds which are isospectral but have different maximal isotropy order? (More generally: Do the results of [DS05] on the spectral determination of the singularities of compact hyperbolic orbisurfaces extend to higher dimensional hyperbolic orbifolds, or are there counterexamples?) Do there exist pairs of flat Riemannian orbifolds which are isospectral on p-forms for all p and have the property just mentioned? Are pairs of flat

Riemannian orbifolds which are isospectral on p-forms for all p necessarily Sunada isospectral?

Another project is the spectral geometry of vector bundles and its relation to geometric quantization. In an ongoing collaboration with Carolyn Gordon and William Kirwin we have constructed pairs of compact, non-symplectomorphic (sometimes not even homeomorphic) Kähler manifolds and pairs of hermitian line bundles over them which are isospectral in all powers (with respect to certain canonically chosen connections) and whose Chern class is the symplectic form on the base manifold. We will work on investigating further and possibly extending these examples in order to take into account the notion of polarizations in the base manifolds (and corresponding restrictions of the Hilbert spaces of sections of the line bundles), which is crucial in geometric quantization theory.

2.2 Cooperations within the SFB

Path integral methods play an important role in most versions of quantum physics. We hope that our results can be useful for projects A1 and A6. Some of the operators studied in project B6 (e.g. [BGR07, GH07b, GH07a, GH07c]) give rise to special field equations similar to the ones studied in project A2. Manifolds with special geometries, which are of interest for some of the problems formulated in this project, provide another relation to project A2. The work on incomplete manifolds has overlap with the aims of B1 even though the methods to be used are different. The spectral geometry of orbifolds which is one of the topics in our project provides another strong link to the investigation of singularities in project B1. In B4 one tries to derive isoperimetric inequalities using flow methods. In this project we intend to use them to gain analytic control on certain operators. Moreover, the work on the geometry of Deligne cohomology classes is mostly intended to provide the necessary basics for some variational problems studied in B4.

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