Introduction

The guiding problem in algebraic geometry is the classification of algebraic varieties up to isomorphism. For varieties of dimension 1 this problem is approached by considering the moduli space $\overline{\mathcal{M}}_g$ of curves of genus g. This space lies at the heart of algebraic geometry and is of enormous interest to algebraic geometers and string theorists. This project proposes the study of several fundamental questions about the global and the enumerative nature of the moduli space of curves.

A good way of understanding the birational geometry of $\overline{\mathcal{M}}_g$ is by describing the cones

$$\mathrm{Ample}(\overline{\mathcal{M}}_g) \subset \mathrm{Eff}(\overline{\mathcal{M}}_g)$$

of ample and effective divisors respectively. The Fulton-Faber conjecture predicts that the dual of the cone $\mathrm{Ample}(\overline{\mathcal{M}}_g)$ is finitely generated by rational curves (1-strata consisting of stable curves with $3g-4$ nodes). This suggests that from the point of view of birational geometry, $\overline{\mathcal{M}}_g$ behaves like a Fano variety even though, for large g, it is a variety of general type! Determining the effective cone of divisors $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ is a much more difficult problem and the shape of the cone is governed by the slope invariant. For an effective divisor $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$ on $\overline{\mathcal{M}}_g$, one defines its slope as the non-negative number

$$s(D) := \frac{a}{\min_{i\geq 0} b_i}.$$

Recall that λ is the class of the Hodge bundle and $\delta_0, \ldots, \delta_{[g/2]} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ are the boundary classes corresponding to different types of singular curves. The slope of the moduli space is the number

$$s(\overline{\mathcal{M}}_g) := \inf_{D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D) \geq 0.$$

The slope $s(\overline{\mathcal{M}}_g)$ determines the Kodaira dimension of $\overline{\mathcal{M}}_g$. The series of counterexamples to the Harris-Morrison Slope Conjecture show that the original prediction $s(\overline{\mathcal{M}}_g) \geq 6+12/(g+1)$ cannot hold, at least for g sufficiently large. We raise the fundamental question of finding lower estimates for $s(\overline{\mathcal{M}}_g)$.

Main projects

1. *Intrinsic coordinates on* $\overline{\mathcal{M}}_g$

We propose a method of constructing explicit systems of coordinates on $\overline{\mathcal{M}}_g$ in the neighbourhood of certain points (0-strata) with the aim of reducing the problem of determining the number of global sections of vector bundles over $\overline{\mathcal{M}}_g$ to counting integer points in certain explicit $(3g-3)$-dimensional polytopes. In particular, this determines when a divisor on $\overline{\mathcal{M}}_g$ is effective or not. The main application of the method is that there exists a constant $\epsilon > 0$ (independent of g!) such that $s(\overline{\mathcal{M}}_g) \geq \epsilon$ for all g. Initial calculations indicate that one can choose $\epsilon = 4$. This estimate would also provide a novel solution to the Schottky problem of distinguishing Jacobians among all principally polarized abelian varieties.

2. *The birational type of Hurwitz spaces and the minimal model of the moduli space of curves*

Using the techniques of Koszul cohomology we plan to study the birational geometry of the Hurwitz schemes $\mathcal{H}_{g:k}$ of k-fold covers of $\mathbb{P}^1$ by genus g curves. The Hurwitz scheme $\mathcal{H}_{g:k}$ establishes an interesting correspondence between the rational moduli space $\overline{\mathcal{M}}_{0,2g+2k-2}$ and the variety of general type $\overline{\mathcal{M}}_g$. We expect that the nature of the Hurwitz scheme changes dramatically around the line $k = g/3$, from uniruledness to varieties of general type. Similar syzygy techniques are employed to find a modular description for the canonical model of $\overline{\mathcal{M}}_g$ using Green's Conjecture on syzygies of canonical curves.

3. *Moduli spaces of higher dimensional varieties*

Despite tremendous progress on the geometry of $\overline{\mathcal{M}}_g$, moduli spaces of higher dimensional varieties have been little studied so far. Having fixed Hilbert polynomials $h_0, \ldots, h_n$, we denote by $\mathcal{M}_{h_0,\ldots,h_n}$ the coarse moduli space of n-log varieties parameterizing objects $(X, B_0, \ldots, B_n)$, where X is an m-dimensional variety, $B_0, \ldots, B_n$ are $\mathbb{Q}$-Cartier divisors on X such that $K_X + B_0 + \cdots + B_i$ is an ample (positive) divisor class having Hilbert polynomial h_i for all $i = 0, \ldots, n$. In dimension 1 these spaces specialize to $\overline{\mathcal{M}}_{g,n}$. We introduce functorial clutching and attaching maps between these spaces $\mathcal{M}_{h_0,\ldots,h_n}$ giving rise to a system of tautological intersection rings for higher dimensional varieties. We study analogues of Faber's Conjecture for the spaces $\mathcal{M}_{h_0,\ldots,h_n}$ as well as develop the theory of geometric divisors for these spaces with the aim of determining the birational type of $\mathcal{M}_{h_0,\ldots,h_n}$. As a first application, we shall consider the moduli spaces of polarized abelian and $K3$ surfaces as well as moduli spaces of plane curves viewed as spaces of log surfaces $(\mathbb{P}^2, C)$.